STUDY OF THE ELECTROSTATIC FIELD WITH OPEN BOUNDARY USING THE FINITE ELEMENT METHOD AND KELVIN TRANSFORMATION

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Abstract: In this paper there is performed a comparative study of the electrostatic field produced by a d.c. voltage applied to a system with 2 extremely long rectilinear conductors, with rectangular cross-section, placed in air. Firstly an experimental study is presented, using an electrokinetic model in an electrolytic tank. Afterward a numerical solution using FEM is presented, (the Kelvin's transformation was used in order to truncate the field domain's boundary). Because the experimental electrokinetic model is also using Kelvin's transformation (geometrical inversion), the comparative study of both methods used to solve the field problem is authorized and makes possible the formulation of some conclusions with respect to the efficiency of both procedures.

Key words: electrostatic field, electrokinetic model, finite elements, Kelvin's transformation

1. EXPERIMENTAL STUDY OF THE ELECTROSTATIC FIELD

1.1. Field problem formulation. Analogies

1.1-a. Field problem formulation. In order to perform this study it was considered, as example for control, a system consisting of 2 parallel, rectilinear and extremely long conductors, placed in air. One performs an analysis of the conductors' electrostatic field, when they are supplied by a d.c. voltage $U = V_1 - V_2$, though analogical (physical) modeling in a tank filled with an electrolytic solution. Finally one calculates the specific capacity of system. From now on we should use the specific term "original" to designate the electrostatic system and respectively the specific term "model" to designate the electrokinetic system from tank.

Problem type. For symmetry reasons, one consider a bidimensional problem for the analysis of electric fields with plan-parallel symmetry, reported to a system of cartesian coordinates (x,y) or to a system of polar coordinates (r,θ). The analysis plane is perpendicular over the conductors.

The real boundary of the field domain is unbounded because the field is extended toward infinite.

The computation boundary is truncated applying Kelvin's transformation. In this transformation a circle is used to enclose the region of interest of the conductors and a second circle is attached to the back of the first circle.

1.1-b. Analogy between the electrostatic field and the steady electrokinetic field. The electrostatic field equations in homogeneous dielectric materials, in the absence of electric charges, and the equations of the steady electrokinetic field in homogeneous conductors are formally identical (Puscasu, 1990; Sora, 1980). As a consequence, one can determine analogies between the electric quantities, the material constants and the global parameters that characterizes both fields, according to table 1. The electrokinetic field quantities from the physical model are denoted by the index "m".

1.2. Kelvin's transformation. Double layered electrolytic tank

1.2-a. Kelvin's transformation (Meeker, 2003; Freeman and Lowther, 1988; Ciric and Wong, 1989). The electrostatic field of the conductors' system is unbounded (its boundary is open). Because the physical model must have finite sizes, it is required to truncate the field domain boundary. For this aim one applies Kelvin's spatial transformation. When this trans-formation is used, the field infinite domain is divided into 2 areas. The first area is finite and represents the field domain of interest, around sources, bounded by a circular boundary with the radius r_0.

Table 1: Analogies

	The electrostatic field ($\overline{E},\overline{D},V$)	**The steady electrokinetic field** ($\overline{E}_m,\overline{J}_m,V_m$)
	A. The equations of the fields	
	1. The first order differential equations	
	$\mathrm{curl}\,\overline{E}=0$,	$\mathrm{curl}\,\overline{E}_m=0$,
	($\overline{E}=-\mathrm{grad}\,V$),	($\overline{E}_m=-\mathrm{grad}\,V_m$),
	$\mathrm{div}\,\overline{E}=\mathrm{div}\,\overline{D}=0$,	$\mathrm{div}\,\overline{E}_m=\mathrm{div}\,\overline{J}_m=0$,
	($\overline{D}=\varepsilon\overline{E}$).	($\overline{J}_m=\sigma_m E_m$).
	2. The Laplace's equations	
	$\Delta V=0$,	$\Delta V_m=0$,
	in cartesian coordinates	
	$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$.	$\frac{\partial^2 V_m}{\partial x^2}+\frac{\partial^2 V_m}{\partial y^2}=0$.
	3.The boundary conditions:	
	a) the general case	
Dirichlet:	$V(P)=f_D$,	$V_m(P_m)=f_{D,m}$.
Neumann:	$\frac{\partial V(P)}{\partial n}=f_N$,	$\frac{\partial V_m(P_m)}{\partial n}=f_{N,m}=-\frac{J_{n,m}}{\sigma_m}$.
Robin:	$aV(P)+b\frac{\partial V(P)}{\partial n}=f_R$,	$a_m V_m(P_m)+b_m\frac{\partial V_m(P_m)}{\partial n}=f_{R,m}$.
	b) the truncation of outer boundaries applying Kelvin transformation	
Periodic:	$V(\bar{r}=\bar{r}_0,\theta)=V(\overline{R}=\bar{r}_0,\theta)$,	$V_m(\bar{r}_0=\bar{r}_{0,m},\theta)=V_m(\overline{R}_m=\bar{r}_{0,m},\theta)$.
Point on boundary:	$P(r)\in\Sigma$,	$P_m(r_m)\in\Sigma_m$.

B. Quantities and materials constants:

electric field strength	$\overline{E}$	$\overline{E}_m$	electrokinetic field strength
electrical displacement	$\overline{D}$	$\overline{J}_m$	electric current density
electrostatic potential	V	V_m	electric steady potential
electrical charge	q	I_m	electric current
electrical charge density	ρ	J_m	electric current density
electrical permitivity	ε	σ_m	electrical conductivity

The electrostatic field		The steady electrokinetic field
	cartesian coordinates of a point:	
(x,y).		$(x_m,\ y_m)$.
	polar coordinates of a point:	
(r,θ)		(r_m,θ_m)
	physical integrals parameters:	
electrical capacitance C		G_m electrical conductance

a, b, f_D, f_N, f_R in the both fields are fonctions of point on boundary

The second region is infinite outside the circular finite boundary. The Kelvin transformation, defined through the geometric relation:

$$R = \frac{r_0^2}{r}\ ,\ r \geq r_0 \qquad (1)$$

performs a transformation of the domain's second infinite area ($r_0 \leq r < \infty$), into a circular finite area, with the radius $R = r_0$, and having in its center ($R = 0$) the counterpart of the point from infinite from the infinite field domain ($r \to \infty$).

Laplace equation in polar coordinates. In the first circular area around sources, the electric potential satisfies the known equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0\ , \qquad (2)$$

After derivations in (1) and substitutions in eq. (2), in the second circular area, the electric potential, $V = V(R, \theta)$, satisfies the equation:

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 V}{\partial \theta^2} = 0\ . \qquad (3)$$

Because the forms of equations (2) and (3) are identical, one can sustain that the second finite circular area without field sources models the infinite area from outside the circle with radius r_0.

The electric potential V, along the boundary with radius $r = R = r_0$, satisfies both eq. (1) and eq. (2). Consequently, along the circular boundaries one must impose a periodic condition that forces the potential's continuity along the corresponding areas from both common boundaries.

In the second circular area it is required that the potential in the center should be zero, $V|_{R=0} = 0$. This value corresponds to the zero value of the potential at infinite, $V|_{r=\infty} = 0$, in the original – untransformed – domain. Under these conditions, the field solution is unique.

1.2-b. Double layer electrolytic tank. The electrolytic tank is represented by an isolated circular tank, filled with electrolyte (Puscasu, 1990). The radius of tank's circular cross-section is r_{0m}, big enough with respect to its height so as to offer the plane-parallel field from model. According to Kelvin's transformation, in the tank filled with a homogeneous electrolytic solution one gets 2 layers of isolated parallel conductors (the isolation can be performed for example by means of a glass sheet whose diameter is a little bit smaller than that of the tank). The circular boundaries of both electrolyte layers are connected near tank's wall, so that the periodicity condition along the boundary is satisfied. The solution of conductive electrolyte from above the glass plate corresponds to the first circular area of the field around conductors, $r_m \leq r_{0m}$. The second electrolyte layer, $0 \leq R_m \leq r_{0m}$, below the glass plate, performs a modeling of the second external, infinite area of the original field.

The electric conductors are modeled within the tank through 2 copper electrodes considered as perfect conductors, through which the electric current is injected. To get a steady electrokinetic regime within the tank, the electrodes supplying should be made by a d.c. source. But the d.c. current should cause the electrolyte polarization. Therefore the supplying is performed from an a.c. source, at low frequency ($f = 1000$ Hz). In this way one gets a regime close to the steady regime.

1.2-c. The reproduction scales original to model. The realization of physical model, corresponding to the original system, is performed for a certain reproduction scale, so as to reproduce both the geometric sizes and respectively the supplying sources values. Consequently, one chooses, depending on the possibilities of practical realization, two reproducing scales: for lengths and respectively for electric voltages.

The scale used for the reproduction of lengths is the constant ratio: (the original system sizes)/ (the model system sizes). This is in cartesian coordinates:

$$k_l = \frac{x}{x_m} = \frac{y}{y_m}\ . \qquad (4)$$

The coordinates for a point from model can be easily determined, from eq. 4, if its coordinates from the original system are the reproduction scale for lengths are known.

The scale used for the reproduction of sources is the constant ratio: (the values of voltage supplying sources from the original system)/ (the values of voltage supplying sources from the model):

$$k_v = \frac{U_{cc}}{U_{m,ca}}\ . \qquad (5)$$

1.2.d. The determination of the electrical quantities from the original is easily performed using the reproduction scales and the values of electrical quantities from the model.

To be more specific:

- *the potential corresponding to a point from the original system* is determined with the relation

$$V(P) = V_m(P_m)\,k_v\ , \qquad (6)$$

- *the electric field strength in a point from the original system* is determined with the approximate relation

$$E(P) \approx \frac{(V_{m1} - V_{m2})k_v}{\Delta l_{m,12}\,k_l}\ , \qquad (7)$$

where: V_{m1} and V_{m2} are potential's values along two equipotential lines, that are closed enough to the calculation point from the model, P_m, and $\Delta l_{m,12}$ is the distance between both equipotential lines;

- *the capacity of the original system*, C_{12}, corresponds to the conductance from the electrokinetic model, G_{m12}:

$$C_{12} = \frac{q_1}{V_1 - V_2},\ G_{m,12} = \frac{I_{m1}}{V_{m1} - V_{m2}},\quad (8a,b)$$

where V_{m1} and V_{m2} are the electrodes potentials, I_{m1} is the current injected in an electrode that is afterward distributed within the electrolyte through the electrode's lateral surface considered as perfectly conducting.

1.3. Experimental schematic and results

1.3-a. Experimental schematic. The electrolyte tank and the experimental schematic, depicted by fig. 1, were realized in the laboratory of Electrical Engineering and the experimental procedure is shown in (Puscasu, 1990).

The elements from the experimental schematic are:
Cv – the tank filled with electrolyte (H2O), in which there are symmetrically placed two identical conducting copper electrodes, C1 and C2;
G – audio-frequency generator, of type H04-002;
R1 and R2 – correlated resistances (R1+R2=const.);
S– probe for the detection of equipotential points in the electrolyte around electrodes.

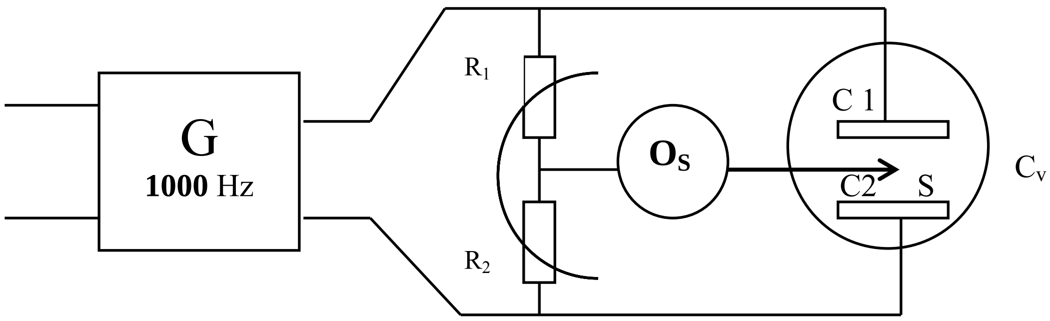


Fig.1. Experimental schematic

1.3-b. Experimental procedure and results. To determine the quantities from model, one must start by drawing the equipotential lines from the electrokinetic field in the electrolyte. Afterwards the electric field strength is determined.
The analogous quantities from the original system are determined with the eq. (7, 8 and 9).

Determination of equipotential lines. The points from an equipotential line are determined by means of a probe S for a certain ratio provided by R_1 and R_2. When the experimental bridge schematic is balanced (the signal recorded by scope is minimum), the coordinates of the point (x_m, y_m) are determined using two rectangular rules. One ruler is fixed to tank, and the second ruler, on which the probe S is placed, is mobile with respect to the first ruler, through a guidance system.
Five equipotential lines are drawn between both electrodes, corresponding to the following ratios of correlated resistances:

$$R_1 + R_2 = \text{const.} = 1000\ \Omega,$$

$$\frac{R_2}{R_1} = \frac{100\ \Omega}{900\ \Omega}, \frac{200\ \Omega}{800\ \Omega}, \ldots, \frac{500\ \Omega}{500\ \Omega}. \quad (9a,b)$$

Then another four lines are drawn, using the symmetry. Therefore one can get the equipotential lines with the following potentials (expressed in percents, 10%, 20%,…, 90%, from the voltage applied on electrodes).

Drawing of electric field lines. The lines corresponding to the electric field strength E are orthogonal on the equipotential lines and can be drawn, with a good accuracy, mainly within the are between the electrodes, where the field is almost uniform. The lines from both field spectra forms curvilinear quadrilaters with almost equal sides,

$$\Delta l_{m,E} = \Delta l_{m,V} = \Delta l_m. \quad (10)$$

For different values of the voltage U that supplies the original system, the electric field strength from the electrostatic system is calculated with the equation (8) that becomes

$$E(P) = \frac{(V_{m1} - V_{m2})\%}{100\,\Delta l_m}\frac{U}{k_1} = 0{,}05\frac{U}{\Delta l_m}, \quad (11)$$

where $\Delta U_{m,12} = (V_{m1} - V_{m2})\% = 10\%$ represents the percent from the voltage applied to the model's electrodes corresponding to a pair of successive equipotential lines, and the reproduction scale for lengths is chosen as $k_1 = 2$.
Experimental results. One draws the equipotential lines spectrum (an example is depicted by fig. 2), and table 2 presents the data measured in the model and respectively those calculated through a graphical-analytic method for the original system.

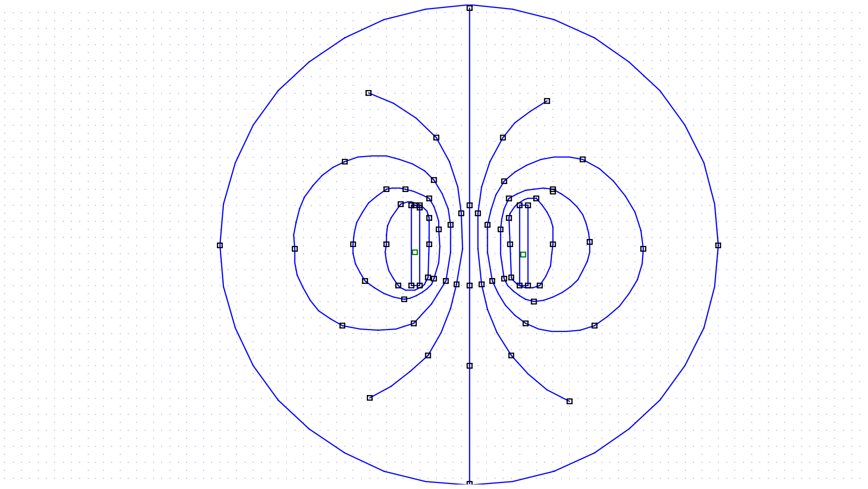


Fig.2. Equipotential lines in electrokinetic model

2. ELECTROSTATIC FIELD ANALYSIS USING FINITE ELEMENTS METHOD

2.1. Formulation of field problem and selection of computation model

The field problem, as specified from the beginning, is a bidimensional problem with plan-parallel symmetry and now is analyzed with the FEM method using the variational formulation (Silvester, 1990; Pei-bai Zhou, 1993). The differential equation of the electrostatic potential (2), using polar coordinates, (r, θ) is substituted by an equivalent system of algebraically equations, obtained through the minimization of the energetic functional with the expression

$$F(V) = \int_{S_1+S_2} \left\{ \frac{\varepsilon}{2} \left[\left(\frac{\partial V}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial V}{\partial \theta} \right)^2 \right] - \rho V \right\} r \, dr \, d\theta, \quad (12)$$

where the potentials of the nodes from mesh is unknowns. The mesh consists of triangular elements, S_1 and S_2 are the surfaces corresponding to both circular areas, ρ is the superficial density of the electric charge from the conductors, and V is the known potential of the conductors from system (S_1).

The field domain is represented by two circular surfaces with equal radius (r_0), determined according to Kelvin's transformation presented above.

The field domain boundary is represented by the circular curves Γ_1 and Γ_2 that bound the plane subdomains used for analysis.

The conditions along the circular boundaries are periodic. Through the computer program one imposes the electric potential continuity along both circular boundaries, each of them consisting of two corresponding semicircles. In the second circular area's center one imposed the condition of null potential. Under these circumstances, the field numerical solution is unique.

2.2. Determination of field problem numeric solution

The FEM solution of field problem is performed using FEMM program (Meeker, 2003). The field domain meshing is based on triangular linear elements. For a convenient mesh step, the field domainwas divided into 24708 elements with 12716 nodes. A finite element model is shown in Fig.3.

The determination of numerical solution and its processing. The computation program provides a solution whose error is at most 10^{-8}. The solution processing, for the objectives of this paper, consists in the drawing of equipotential lines (Fig. 4) and the calculation of system's specific capacity using the relation:

$$C_{12} = \frac{q_1}{V_1 - V_2} = \frac{q_1}{U}, \quad (13)$$

where for a known applied voltage, U_1, one determines numerically the total electrical charge q_1, distributed along the first conductor surface.

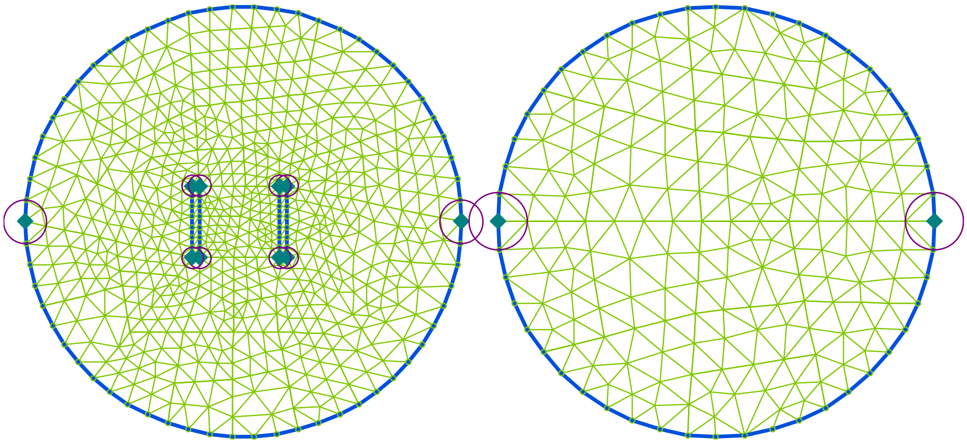
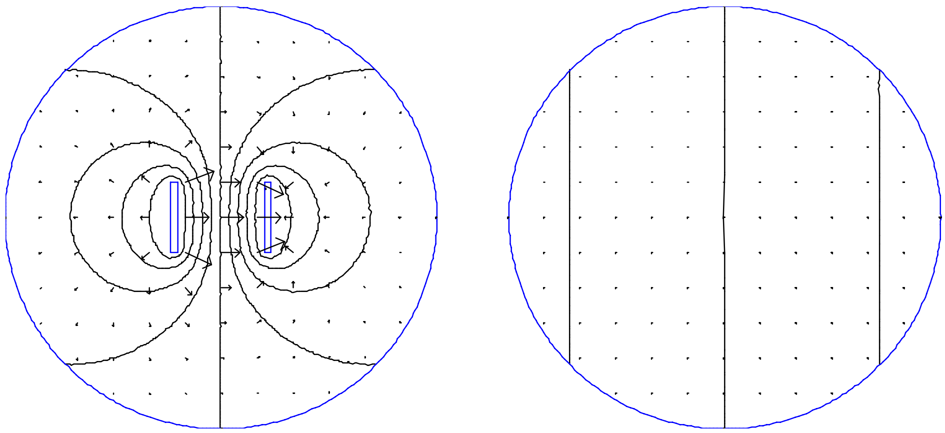


Fig.3. Discretization of field domain with triangular elements



$0 \le r \le r_0$ $\quad$ $0 \le R \le r_0$, respectiv $r_0 \le r < \infty$,

Fig.4. Equipotential lines and electric field strength lines distributions

3. RESULTS

3.1. Electric field strength

To make the comparative study, for the beginning one represents 9 equipotential lines, as shown in the paragraph 1.3-b. Then one determines in a graphical-analytical manner, the electric field strength in 9 points, carefully selected along the mean field line, between both electrodes. These points correspond to the 9 equipotential lines from the electrokinetic field.

The values from the experimental calculation, E_e, considered with respect to the distances Δlm measured between the neighboring equipotential lines are presented by Table 2 along with the numerically calculated values, E_n. Figure 5 depicts the electric field strength variation, numerically determined along the mean line between the conductors from the original electrostatic field.

3.2. Specific capacitance of the electric line

Specific capacitance of the electric line will be determined, in a future paper, through physical modeling in the electrolytic tank and numerical calculation with the finite element method on the electrostatic and electrokinetic numerical models.

4. CONCLUSIONS

1. The experimental results are affected by errors caused by the realization of the analogue model, the measurements using the model, the boundary truncation and – most of all – the graphical-analytical calculation. The overall error generated by them can significantly affect the accuracy corresponding to the determination of the local state quantities and of specific capacity. The errors generated by the graphical-analytical calculation can

be reduced through the drawing of a large number of equipotential lines, but this should make the procedure almost unusable.

2. Despite of all its disadvantages, the physical modeling of technical systems enables the obtaining of some calculation information useful in design.

3. Today the numerical modeling with FEM takes the advantages of performing programs that provide solutions for many complex problems with a significant accuracy.

4. The truncation of open borders with Kelvin's transformation represents a good procedure that leads to the reducing of the required memory and of computation time, for an imposed accuracy with the order 10^{-8}.

5. A study on a field problem with infinite boundaries made through physical modeling and numerical calculations enables the obtaining of knowledge on making experiments and numerical simulations. It also provides abilities on the selection of methods of analysis that require small costs and high accuracy.

Table 2: Values of electric field strength determinated experimental and numeric

The coordinates of points (x,y) [cm]	(-10,0)	(-8,0)	(-6,0)	(-4,0)	(0,0)	(4,0)	(6,0)	(8,0)	(10,0)
0	1	2	3	4	5	6	7	8	9
Δl_m [cm]	1,2	1,2	1,3	1,3	1,3	1,3	1,3	1,2	1,2
E_e [V/m]	83,33	83,33	77	77	77	77	77	83,3	83,3
E_n [V/m]	86,72	84,76	82,11	80,08	79,4	80,10	82,09	84,36	86,87

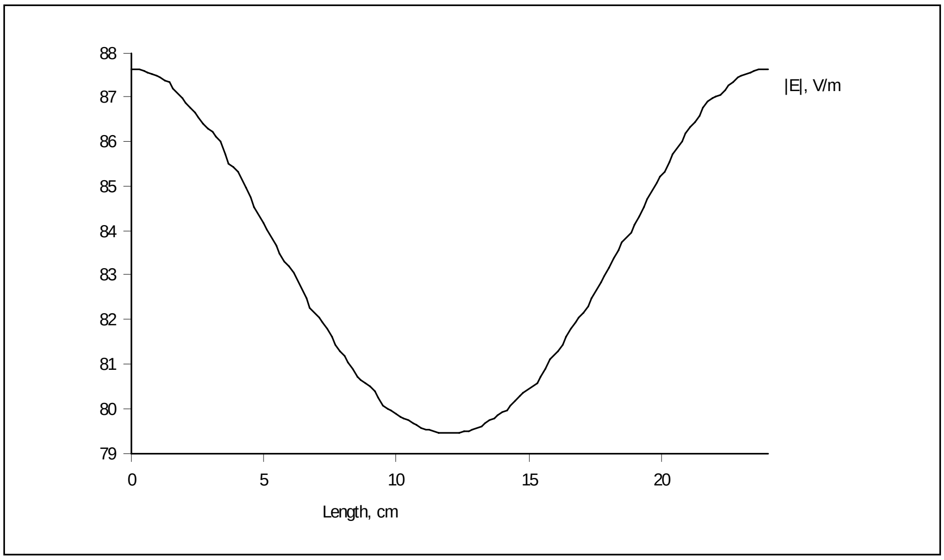


Fig.5. Electric field strength along of the field middle line between conductors

Acknowledgement

The author wants to thank to his collegues for the discussions and its support about this study.

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