A GENERAL APPROACH TO THE DECOMPOSITION OF DYNAMIC PROGRAMMING BY PROGRAMMING FOR NONLINEAR CONTROL SYSTEMS

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Abstract **- A new approach to the solution of nonlinear programming problems is suggested. The main salient feature of the approach is a network interpretation of nonlinear functions. Convergence rate conditions are considered for sequential conventional numerical dynamic programming algorithm for equality and inequality constrains. An important advantage of the technique proposed is that it is particularly suitable for parallel processing, and the total computational time can be reduced in a factor proportional to the number of time intervals provided that the quantity of the processor for parallel processing is enough.**

Keywords: *dynamic programming, decomposition, nonlinear programming, optimal control, parallel processing.*

1. INTRODUCTION

It is well-known that a class of dynamical systems without control modeled by the deterministic nonlinear difference equation often exhibits chaotic oscillations [1]:

$$\sum_n \begin{cases} x_{n+1} = Ax_n(1-x_n-y_n); & (1)\\ y_{n+1} = By_n(-1+\dfrac{x_n}{c}) - H. & (2)\end{cases}$$

$$\sum_e \begin{cases} x_{n+1} = Ax_n(1-x_n-y_n); & (4)\\ y_{n+1} = B(1-E)y_n(-1+\dfrac{x_n}{c}). & (3)\end{cases}$$

and

$$\sum_u \begin{cases} x_{n+1} = Ax_n(1-x_n-y_n); & (5)\\ y_{n+1} = By_n(-1+\dfrac{x_n}{c}). & (6)\end{cases}$$

where x_n and y_n represent the population densities of preys and predator at the *n*-th generation, respectively, *A* and *B* rates positive constant representing the growth rates of the prey and predator respectively; *C* is a positive constant representing the interconnected rate between two species; *H, E* constant control [2].

A lot of optimization problems can be formulated by a deterministic finite stage dynamic programming model. It is more frequent that such a dynamic programming formulation does not have a closed form solution. For dynamic programming problems where the state variables are of the high dimension the conventional numerical dynamic programming algorithm is in fact applicable owing to the well known differently in relation to the "curse of dimensionality".

Let *S={I, U}* a finite oriented network without loops, *I, U* are sets of nodes and arcs, respectively a nod $i \in I$ has a finite number of inputs and outputs. The input signals (flows) of the nods *i* are summed up, and the copies of the sum are passed to every output of the node. A nod $j \in I$ is called a source of the networks *S* if one of its input signal is not an output one of the other nodes. A node $t \in I$ is called a sink if one of the output signal does not enter the other nodes of the network [3]. Let *S, T* are sets of sources and sinks, respectively. An ark $(i,j) \in U$ transfers the signal z_i from the node $i \in I$ to the node $j \in I$ in the form $f_{ij}(z_i)$, $f_{ij}(z)$ is a characteristic of he arc. Signals z_t, $t \in T$ are single – valued functions with respect to the variables x_j, $j \in I$. If the characteristics $f_{ij}(z)$, $(i,j) \in U$ are smooth function, then the problem

$$z_i \to \min, z_i = \sum_{j \in I} f_{ij}(z_j) + \sum_{s \in I} b_{is} x_i,\ i \in I, \quad (7)$$

$$z_i = 0,\ t \in T_0,\ d_{*j} \le x_j \le d_j^*,\ T_0 = T/c,$$

c is the node from *T*, is a smooth nonlinear programming problem. It is easy to prove that

$$\eta_i = \sum_{j \in J} X(i,j)\xi_j,\ i \in I, \quad (8)$$

$$\frac{\partial \varphi_i(x_j)}{\partial x_s} = X(t,s),\ \ t \in T,\ s \in I. \quad (9)$$

The network *S* is called locally controllable on the flow *z* if for any collection $\eta = (\eta_t, t \in T_0)$ the exists such a collection $\xi = (\xi_j, j \in J)$ that

$$\varphi_i(x + \theta\xi) = \varphi_i(x) + \theta\eta_t + O(\theta),\ t \in T_0.$$

The criterion of local controllability is the condition rank $X(T_0, I/x) = |T_0|$. Connect with this notion the main constructions of the approach:

- a set $I_{\sup} \subset I, |I_{\sup}| = |T_0|$, is called a local support of *S* on *z*, if $\det P \ne 0, P = P(x) = X(T_0, I_{\sup}/x)$;

- a pair $(x, I_{\sup})$ of feasible input *x* and support I_{sup} is input;

- a support input is nondegenerate if $d_{*j} \le x_j \le d_j^*,\ j \in J_{\sup}$;

- the vector $U' = U'(x) = (U_t(x), t \in T_0)' = -X(t_0, I_{\sup}/x) \cdot P^{-1}$ is called a vector of potentials.

A network $S^*=\{I^*,U^*\}$ is conjugate to S_t if it is constructed by altering the current:

- the sources of S^* are the sinks of S_t;
- the sinks of S^* are the sources of S_t;
- the arc $(i,j) \in U^*$ is obtained from the arc $(i,j) \in U_t$ by altering current;
- the nodes $i \in I^* = I$ of the networks S^* act as the nodes of S_ξ.

A unified description of the networks S, S^* is achieved using the function

$$H(z,\psi,x) = \sum_{i \in I} \psi_i \Big(\sum_{j \in I(i)} f_{ij}(z_j) + \sum_{s \in I} b_{is} x_s \Big);$$

$$z_i = \frac{\partial H}{\partial \psi_i},\ i \in I;\ \psi_i = \frac{\partial H}{\partial z_i},\ i \in I/T; \tag{10}$$

$$\varphi_t = \frac{\partial H}{\partial z_t} + v_t, t \in T,$$

v_t is input signal of S^*, $\psi = (\psi_i, i \in I)$ are the corresponding output signals of nods $i \in X_t^*, b_{tt}^* = 1, b_{it}^* = 0,$ if $i \ne t, t \in T, b_{it}^* = 0,$ if $i \notin T, t \in T$.

For optimality of the feasible input x^0 it is necessary that there exist a support I_{sup} that along $\{x^*, I_{sup}\}$ and the accompanying flow z^0 an coflow ψ^0 the condition $H(z^0, \psi^0, x^0) = \max H(z^0, \psi^0, x), d_* < x < d^*$ is fulfilled. The key role in this result belongs to the support.

2. THE SECOND – ORDER OPTIMALITY CONDITIONS

Let x^0 be an optimal input,

$$I_* = \{j \in I, x_j^0 = d_{*j}\}, I^* = \{j \in I, x_j^0 = d_j^*\},$$

$$I^0 = \{j \in I, d_{*j} < x_j^0 = d_j^*\}.$$

The totality $\{I_*, I^*, I^0\}$ is called a structure of optimal input x^0. The algorithm consists of the main part and the finishing procedure [5].

The main part of the algorithm reveals the structure of the optimal input and prepares the initial guess for the finishing procedure which constructs x^0 to a required accuracy.

With appropriate initial support input $(x, I_{\sup})$ on which the conditions for $\alpha > 0$ sufficiently small

$$\psi_j \le \mu(\alpha),\ x_j \ge d_j^* - \mu(\alpha), j \in I^*;$$

$$\psi_j \le -\mu(\alpha),\ x_j \ge d_{*j} + \mu(\alpha), j \in I_*; \tag{11}$$

$$|\psi_j| \le \mu(\alpha), d_{*j} < x_j < d_j^*, j \in I_s; |z(t)| \le \mu(\alpha),$$

$$t \in T_0;\ I_* \cup I^* \cup I_\xi = I_n,\ D(I_s, I_\xi) > 0,$$

$\mu(\alpha) \to 0, \alpha \to 0$ are fulfilled, one can determine the structure of the optimal input and construct x^0 using the finishing procedure directly. The finishing procedure consists in solving the system of nonlinear equations

$$z_i(\chi) = 0, t \in T;\ \psi_j(x) = 0, j \in S_f,$$
$$\chi_j = d_j^*, j \in I^*; x_j = d_{*j}, j \in I_*, \tag{12}$$

under variables $x_j, j \in I_f \cup I_{\sup}$, by the Newton method using the initial guess $\chi_j = x_j,\ j \in I_j \cup I_{\sup}$. Let χ^0 be a solution of the system (12) for which

$$\psi_j(\chi^0) \ge 0\ at\ j \in I_*; \psi_j(\chi^0) \le 0\ at \tag{13}$$
$$j \in I^*;\ D(I_f, I_f) > 0.$$

Then the χ^0 is a local optimal input.

A network $S_l^\alpha = S_l^*(z) = \{I, U\}$ is called α approximation of S on z_{l*} if

- The characteristics of its arcs $f_{ij}^{l\alpha}(\eta / z_i)$ are linear.
- The nodes have capacities

$$\alpha_{*i} = \alpha_{*i}(\alpha) \le 0; \alpha_i^* = \alpha_i^*(\alpha) \ge 0, i \in I,$$
$$\alpha_i^* = \min_{j \in I^*(i)} \alpha_{ij}^*(\alpha),\ \alpha_{ij}^* = \max_{j \in I^*(i)} \alpha_{ij}^*(\alpha). \tag{14}$$

- The deviation between the functions

$$f_{ij}(z_i + \eta); f_{ij}^{l\alpha}(\eta / z_i); \alpha_{*ij} \le \eta \le \alpha_{ij}^*, i, j \in U, \tag{15}$$

in the given norm does not exceed α. A first-order support problem is as follows:

$$\eta_c \to \min, \eta_i = \sum_{j \in I(i)} f_{ji}^{l\alpha}(\eta_j) + \sum_{s \in I} b_{is} \xi_s, i \in I,$$
$$l_t = 0, t \in T_0; \alpha_{i*} \le \eta_i \le \alpha_i^*, i \in I, \tag{16}$$
$$d_{*j} - x_j \le \xi_j \le d_j^* - x_j, j \in I.$$

Note that the linear approximation (16) of the problem (7) does not use artificial norm conditions. At minimization of complicated functions, serious difficulties arise due to the optimal conditions equalities $\psi_i = 0$. In such a situation, to improve the algorithm, the principle of accumulation of linear approximations of that conditions is suggested. The realization of this principle leads use to a second –order support problem which is obtained by adding to (16) for the linear constraints

$$\psi_j^l(\xi / x) = \psi_j(x) + \xi' \frac{\partial \psi_j(x)}{\partial x} = 0,$$
$$j \in I_{fj},\ I_\xi \subset I_n, D(I_f, I_f) > 0. \tag{17}$$

3. THE AVERAGING METHOD

Consider the following external problem with quickly changing parameters [7]:

$$c'x(t^*) \to \max, \dot{x} = A(t,t/\mu)x + B(t)U, \quad (18)$$

$x(t_*) = x_*, Hx(t^*) = g; |U(t)| < 1, t \in T = [t_*, t^*]$. Here $x \in R^n, g \in R^m, U \in R$, rank $H = m < n$, $\mu > 0$ is the small parameter, $A(t,\tau+1) = A(t,\tau)$, $|U(t,\mu)| < 1$, $t \in T$ is called an s-admissible control if along the corresponding trajectory $x_s(t,\mu), t \in T$. The asymptotic equality $\| Hx_*(t^*,\mu) - q \| = c(\mu^*)(c'x^0(t^*,\mu) - c'x_*(t^*,\mu)) = c(\mu^*)$ is fulfilled, where $x^0(t^*,\mu)$ is the optimal trajectory [6]. The suggested scheme of construction of s-optimal control is based on the expansion of the fundamental matrix

$$F_\mu(t,\tau), \frac{\partial F_\mu(t,\tau)}{\partial \tau} = -F_\mu(t,\tau)A(\tau,\tau/\mu),$$
$$F_\mu(t,\tau-0) = E.$$

It consists of the following steps

1. From base problem

$$c'x_0(t^*) \to \max, \dot{x}_0 = A_0(t)x_0 + b(t)U, \quad (19)$$
$$x_0(t^*) = x_0, Hx_0(t^*) = g; |U(t)| < 1, t \in T.$$

2. Find switching points of the s-optimal control in the form $t_{ij}(\mu) = t_{ij} + \mu t_{ij} + \ldots + \mu^s t_{ij}, j \in I_{\sup}$, the support matrix $P^s_{\sup}(\mu) = (P(t_{ij}(\mu), j \in I_{\sup})$, the vector of potential

$$y_i'(\mu) = (c(t_{ij}(\mu)), j \in I_{\sup}) \cdot \left[P^*_{\sup}(\mu)\right]^{-1} =$$
$$= y_i' + \mu y_i' + \ldots + \mu^s y_i' = (c(t) =$$
$$= c'F_0(t^*,t)b(t), t \in T),$$

the solution of the conjugate system

$\dot{\psi} = -A_0'(t)\psi, \psi_s(t^*,\mu) = H'y_s(\mu) - c$, the control $\Delta_s(t,\mu) = y_s'(t,\mu)b(t) = \Delta_0(t) + \mu\Delta_0(t) + \ldots + \mu^s\Delta_0(t), t \in T$ the quasi-control $U_s(t,\mu) = sign\Delta_s(t,\mu), t \in T$, and the trajectory $x_s(t,\mu)$.

3. Find the number $t_{ij}, i = \overline{1,s}, j \in I$, by the extension of the function $Hx_0(t^*,\mu)$ into Taylor' series in terms of μ up to S order. This results to a systems of linear equations with respect to variables $t_{ij}, j \in I_{\sup}, i = \overline{1,s}$. The matrix of the system is

$$(zp(t_{ij}) sign \dot{\Delta}_0(t_{ij}), j \in I_{\sup}). \quad 1.$$

4. The function $U_s(t_\lambda,\mu) = sign \dot{\Delta}_0(t_{ij})$, $t \in T$, $j \in I_{\sup}$, $i = \overline{1,s}$ is an optimal control of problem (19).

Consider the following problem with quickly changing input device:

$$c'x(t^*) \to \min, \dot{x} = A(t)x + b(t,t/\mu)U, \quad (20)$$
$$x(t_0) = x_0, Hx(t^*) = q; |U(t)| \le 1,$$
$$b(t,\tau+1) = b(t,\tau).$$

The scheme of solution of problem (20) is analogues with the above one.

A general problem

$$c'x(t^*) \to \max, \dot{x} = A(t,t/\mu)x + b(t,t/\mu)U, \quad (21)$$
$$x(t_0) = x_0, Hx(t^*) = q; |U(t)| \le 1, t \in T,$$

is solved using the synthesis of the above methods.

4. MULTICRITERIA OPTIMIZATION

In order to describe the extension of the generalized time interval iteration algorithm (TIIA) dealing with multicriteria dynamic programming problems some notations introduced: the set of Pareto control sequences and that of Pareto trajectories obtained by the f_i-th subproblem through fi-th subproblem in the i-th iteration will be described by $U(i,j_1,j_2)$ and $X(i,j_1,j_2)$ respectively. New the problem formulation is the same as that expended by

$$I = \sum_{k=0}^{N-1} V_k(x^k,u^k) = \min, \quad (22)$$

object to
$$x^{k+1} = T_k(x^k,u^k); \quad (23)$$
$$k = \overline{0,N-2};$$
$$n_k(x^k,u^k) = 0; \quad (24)$$
$$g_k(x^k,u^k) \ge 0, \quad (25)$$

except that the functions $V_k, k = \overline{0,N-1}$ a vector valued functions of l –dimension, this new problem will be denoted by VDOCP, for VDOCP, the relevant nonlinear programming problem introduced in the generalized TIIA to be described in the following by VNLP and the operation to find the Pareto solution of the problems will be described by $V_{\min}$. The procedure of the generalized TIIA for the VDCOP is described by

1. The procedure of the generalized algorithm for DCOP ((22)-(25)).
2. At the i-th iteration, where the update of the Pareto solution set pair is *(U(i),X(i))* solve the following s – stage subproblem.

Theorem 1. For the model VDOCP suppose:

1. The hypotheses *H1* and *H2* hold.
2. The set ASP (admissible set pairs) is compact.
3. A type of constraint qualification for Kuhn-Tacher theorem is satisfied for any solution of *VNLP(i,j)* and any $(U,X) \in M$.

Theorem 2. For the model VDOCP suppose:

1. All the components of the vector valued function V_k are strictly convex.
2. All the components of functions $h_k S$ are linear, those of $g_k S$ are concave.
3. The hypothesis *H2* holds.
4. The condition:

- The function *I* is pseudoconvex, h_k both quasiconvex and quasiconcative;
- g_k quasiconcative for $k = \overline{0, N-1}$;
- the set ASP $\{U, X \mid I(U, X) \le \varepsilon, (U, S) \in ASP, \varepsilon \in R\}$

hold.

Then the sequence *(U(i),X(i))* generated by the algorithm converge to the Pareto optimal solution set pair.

Remark:

H1. For any admissible solution pair *(U',X')*, if *U(i)=U', X(i)=X'* then each of the subproblem *NLP(i,j)*, $j = \overline{0, l}$ has *S* unique solution. The optimal control problem formalized by (22)-(25) has also unique optimal trajectory.

H2. For any $x^k \in X, k = \overline{0, N-2}$ summing the *n* functions $T_{kj}(x^k, u^k), j = \overline{1, n}$ there exist *m* functions which can be assumed to be those corresponding to $j = \overline{1, m}$ without lows of generality, the matrix

$$\begin{bmatrix} \frac{\partial T_{k1}}{\partial u_1^k} & \cdots & \frac{\partial T_{k1}}{\partial u_m^k} \\ \cdots & \cdots & \cdots \\ \frac{\partial T_{km}}{\partial u_1^k} & \cdots & \frac{\partial T_{km}}{\partial u_m^k} \end{bmatrix}.$$

in of full rank over the constraint set.

5. CONCLUSION

The most part of suggested algorithms was realized problems (18), (20), (21) a considered as canonical optimal control problems for the averaging method. They are of great importance in optimization of long-time processes. The generalized TIIA provides a powerful tool to tackle large scal dynamic programming problems since it reduces the computer storage thoroughly.

The hypothesis witch ensure the convergence of the algorithm are rather mild which extend the range of applications to include those optimal control problems of practical importance. A distinct feature of the algorithm is that it is flexible in choosing the structure of the decomposition which fits the structure of the problem in hand and hence speed up the convergence, moreover, it is easy to implement on parallel processors with a suitable choice of architecture.

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