6TH INTERNATIONAL CONFERENCE ON ELECTROMECHANICAL AND POWER SYSTEMS
October 4-6, 2007 - Chişinău, Rep.Moldova

METHOD OF BOUNDARY INTEGRAL EQUATIONS IN THE INVESTIGATION OF TRANSIENT HEAT SPREAD IN A CONDUCTOR

Boris DRAGANOV*, Gleb DRĂGAN**

*Ukraine Academy, **Romanian Academy, gleb_dragan@yahoo.com*

***Abstract* – The temperature of heating of a conductor can reach significant values, and, in such cases, the main parameters of the process of heat spread change with time. In the solution of problems with variable parameters, it is expedient to use the method of boundary integral equations (BIE).**

Keywords: *heat, transient transfer, integral equation.*

1. INTRODUCTION

There are two approaches to considering the time effects. One of them consists of taking time into account explicitly, in the same way as spatial coordinates, and carrying out numerical integration over a time interval just as over the geometric boundary of the body. Such a method is applied in [1]. Another approach used in the BIE method is connected with excluding time from the list of independent variables by means of application of the Laplace transformation to the initial partial differential equations and boundary conditions [2]. In such a way, parabolic and hyperbolic differential equations, as a rule, can be reduced to more convenient elliptic equations, which are solved by the BIE method for every element from the sequence of values of the transformation parameter. The solution as a function of time can be obtained afterwards by means of numerical inverse Laplace transformation.

2. APPROACH

Although the BIE method can be applied for the solution of both plane and three-dimensional problems, we consider in the present paper a problem in the plane formulation. For further simplification, we omit the discussion of effects attributable to the presence of volume forces and nonlinear initial conditions.

Consider the process of heat spread in the homogeneous isotropic body of a conductor

$$\frac{\partial T}{\partial t}(x,t) = a\nabla^2 T(x,t), \qquad (1)$$

where T is the temperature, t is the time, x is an arbitrary point of the body, $a = \lambda / c\rho$ is the thermal diffusivity, λ is the thermal conductivity, c is the specific heat, ρ is the density, and ∇^2 is the Laplace operator.

The solution of (1) describing two-dimensional heat spread from a point source of unit intensity, which begins to act at the zero moment of time at a certain arbitrary point x', has the form

$$T(x,t) = f(x',x,t) \equiv \frac{1}{4\pi a t} e^{-\wp^2/4at}, \qquad (2)$$

where $\wp$ is the distance between the points x and x'.

Applying the Laplace transformation to (1), we find

$$a\nabla^2 T^*(x,s) - sT^*(x,s) = 0, \qquad (3)$$

where s is the transformation parameter and T^* is the Laplace transform of temperature T. Equation (3) is satisfied by the Laplace transform of function (2), namely,

$$T^*(x,s) = f^*(x',x,s) \equiv \frac{1}{2\pi a} K_0\left(\sqrt{s/a}\,\wp\right), \qquad (4)$$

where K_0 is the modified Bessel function of second kind and zero order. Applying the second Green formula to f^* and a fairly smooth solution T^*, we reduce (3) to the following equation:

$$T^*(x,s) = a\int\left[\frac{\partial}{\partial n}T^*(P,s)\,f^*(x,P,s) - T^*(P,s)\frac{\partial}{\partial n}f^*(s,P,x)\right]dS, \qquad (5)$$

where Γ is the boundary of a plane domain, S is the distance along Γ, P is a point belonging to Γ, and n is the external normal to Γ at the point P.

If T^* and $\partial T^* / \partial n$ are known everywhere on Γ for any given value of s, then the quantity T^* at an arbitrary internal point x can be determined for this s from relation (5) with the help of a simple quadrature.

However, only a certain part of the boundary values is known beforehand in a boundary-value problem. Meanwhile, relation (5) can also be used for determining the rest of boundary values. This can be achieved by means of passage to the limit in (5) as x tends to an arbitrary point X of the boundary Γ:

$$T^*(X,s)=2a\int_{\Gamma}\left[T^*(P,s)\frac{\partial}{\partial n}f^*(X,P,s)-\right.$$
$$\left.-\frac{\partial}{\partial n}T^*(P,s)\,f^*(X,P,s,P)\right]dS=0. \quad (6)$$

The improper integrals in (6) are understood here in the sense of their principal values. If boundary conditions corresponding to a correctly stated problem are assigned, then we may consider (6) as a singular integral equation with respect to T^* or $\partial T^*/\partial n$ on those parts of the boundary where one of these quantities is not set. The solution of this equation is realized by approximate numerical methods. After finding the unknown values on the boundary, the transform of temperature for any internal point x can be found from relation (5) with the help of simple integration. The final stage in the algorithm of solution consists of carrying out the inverse transformation of the corresponding functions.

For the solution of the problem of transient heat conduction, approximate computational algorithms have been developed. The most widespread approach to the solution of singular integral equations consists of dividing the boundary Γ of the body into a certain number of segments. From the mathematical viewpoint, this corresponds to the presentation of (6) in the form

$$T^*(X,s)=2a\sum_{\xi=1}^{N}\int\left[T^*(P_\xi,s)\frac{\partial}{\partial n}f^*(X,P,s)-\right.$$
$$\left.-\frac{\partial}{\partial n}T^*(P_\xi,s)\,f^*(X,P,s,P)\right]dS=0, \quad (7)$$

where each value of the subscript ξ refers to one of N segments Γ_ξ. It should be noted that the way of division of the boundary is to a certain degree arbitrary and is chosen proceeding from the necessary accuracy. Furthermore, equation (7) does not contain any approximations.

In order to reduce the integral equation (7) to a system of algebraic equations, one has to approximate the boundary variables $T^*(P,\ s)$ and $\partial T^*(P,s)/\partial n$ on each segment with the help of functions of S of a certain given form. For example, $T^*(P,\ s)$ and $\partial T^*(P,s)/\partial n$ can be approximated by polynomials of S of a certain given power, e.g., $T^*(P,\ s)$ can be presented on the ξth segment by a constant $T^*(P_\xi,\ s)$. If, consecutively on each of the segments, we ascribe to the point X a discrete value X_η, (7) changes into a system

$$T^*(X,s)+2a\sum_{\xi=1}^{N}\left[T^*(P_\xi,s)\frac{\partial}{\partial n}\int_{\Gamma_\xi}\frac{\partial}{\partial n}f^*(X_\eta,P,s)dS-\right.$$
$$\left.-\frac{\partial}{\partial n}T^*(P_\xi,s)\int_{\Gamma_\xi}\frac{\partial}{\partial n}f^*(X_\eta,P,s)\right]dS=0;\ \eta=1,2,...,N. \quad (8)$$

These equations form a system of N linear algebraic equations with respect to $T^*(P_\xi,\ s)$ and $\partial T^*(P_\xi,s)/\partial n$. The integrals in (8), which can be calculated by means of numerical integration, represent here the known coefficients at these variables. If the values of variables corresponding to the known boundary conditions of a correctly stated boundary-value problem are assigned, then we can determine the rest of values from the solution of algebraic equations.

References

[1] Chang Y. P., Kang C. S., and Chen D. J., *The use of fundamental Green's functions for the solution of problems of heat conduction in anisotropic media*, Int. J. Heat Mass Transf., Vol. 16, P. 1905 – 1918 (1973).

[2] Rizzo F. J. and Shippy D. J., *A method of solution for certain problems of transient heat conduction*, AIAA J., Vol. 8, P. 2001 – 2009 (1970).