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CONSIDERATIONS ON THE STABILITY OF NONLINEAR SYSTEMS. PART. I: INCREMENTAL INPUT DESCRIBING FUNCTIONS

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Abstract – **The stability of nonlinear systems is an important issue of dynamic systems. The analysis complexity for this category of systems has led to the development of some analytical and graphical-analytical methods.**
The paper emphasizes one graphical-analytical method of analyzing the stability of a nonlinear feedback system, utilizing the method of harmonic linearization. We use the concept of incremental input describing function to characterize the nonlinearity at the input of which we apply two harmonic signals with the same frequency but with much different amplitudes. Through an adequate graphical representation it is possible to appreciate the stability or the instability of the output forced oscillation of the system.

Keywords: *nonlinear system, incremental input describing function, forced oscillations.*

1. THEORETICAL CONSIDERATIONS

Let us consider a nonlinear system with the structure in Fig.1, containing one nonlinear element N and one linear element L with a low-pass filter characteristic.

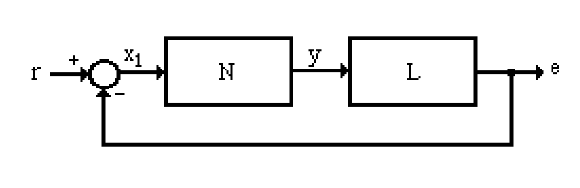

Figure 1: Nonlinear feedback system.

At the input of the system we have the harmonic signal:

$$r(t) = R\sin \omega t \quad (1)$$

Let us suppose that as a result of some perturbation at the input of the nonlinear element, over the component with the frequency ω:

$$x_1(t) = A\sin \omega t \quad (2)$$

one supplementary component is added:

$$x_2(t) = a\sin(\omega t + \varphi) \quad (3)$$

of the same frequency but with different amplitude and phase.
In this way, the input of the nonlinear element becomes the superposition of two harmonical signals:

$$x(t) = A\sin \omega t + a\sin(\omega t + \varphi) \quad (4)$$

or:

$$x(t) = (A + a\cos\varphi)\sin \omega t + (a\sin\varphi)\cos \omega t$$

With the notations:

$$A_1 = A\cos\varphi$$
$$A_2 = A\sin\varphi$$

the previous equation becomes:

$$x(t) = A_1 \sin \omega t + A_2 \cos \omega t$$

and, after further processing:

$$x(t) = X\sin(\omega t + \phi) \quad (5)$$

where:

$$X = \sqrt{A_1{}^2 + A_2{}^2}, \qquad \phi = a\tan\frac{A_2}{A_1} \quad (6)$$

Equation (5) shows that the harmonic oscillation from the input of the nonlinear element has the same frequency as the input harmonic signal but different amplitude and phase, (6).
Let us consider that the amplitude of the component $x_2(t)$ of the output signal caused by the perturbation, a, is much smaller than the amplitude A, $a \ll A$, and thus the following approximations can be made:

$$X = \sqrt{A_1{}^2 + A_2{}^2} = \sqrt{(A + a\cos\varphi)^2 + (a\sin\varphi)^2} \cong$$
$$\cong \; A + a\cos\varphi$$

$$\phi = a\tan\frac{A_2}{A_1} = a\tan\frac{a\sin\varphi}{A + a\cos\varphi} \cong$$
$$\cong a\tan\frac{a}{A}\sin\varphi$$

In this case, equation (5) may be written:

$$x(t) \cong (A + a\cos\varphi)\sin(\omega t + a\tan\frac{a}{A}\sin\varphi) \qquad (7)$$

We note that the signal (7) is applied to the input of the nonlinear element and therefore the fundamental component of the output signal of this element is:

$$\begin{aligned} y(t) \cong\ & Y_1 \sin(\omega t + a\tan\frac{a}{A}\sin\varphi) + \\ & + Y_2\cos(\omega t + a\tan\frac{a}{A}\sin\varphi) \end{aligned} \qquad (8)$$

where:

$$Y_1 = (A + a\cos\varphi)\left[N_R(A + a\cos\varphi)\right] \qquad (9)$$

$$Y_2 = (A + a\cos\varphi)\left[N_I(A + a\cos\varphi)\right] \qquad (10)$$

and

$$N(A) = N_R(A) + jN_I(A) \qquad (11)$$

is the describing function of the nonlinearity of the nonlinear element, N.

Considering $\frac{a}{A} << 1$, the following approximations can be made regarding equation (8):

$$\sin(\omega t + \frac{a}{A}\sin\varphi) \cong \sin\omega t + \frac{a}{A}\sin\varphi\cos\omega t \qquad (12)$$

because:

$$\sin\omega t\ \cos(\frac{a}{A}\sin\varphi) \cong \sin\omega t$$

$$\cos\omega t\ \sin(\frac{a}{A}\sin\varphi) \cong \frac{a}{A}\sin\varphi\ \cos\omega t$$

and correspondingly,

$$\begin{aligned} & \cos(\omega t + \frac{a}{A}\sin\varphi) \cong \\ & \cong \cos\omega t + \frac{a}{A}\sin\varphi\cos\omega t \end{aligned} \qquad (13)$$

because:

$$\cos\omega t\ \cos(\frac{a}{A}\sin\varphi) \cong \cos\omega t$$

$$\sin\omega t\ \sin(\frac{a}{A}\sin\varphi) \cong \frac{a}{A}\sin\varphi\ \sin\omega t$$

At the same time, because the nonlinear element's input signal contains two harmonic components, we can use the method of determining the two-sinusoid-input describing function (TSIDF) by power series expansion, [1] and thus equation (8), along with (12) and (13,) may be written in the following form:

$$\begin{aligned} y(t) \cong\ & Y_{11}(\sin\omega t + \frac{a}{A}\sin\varphi\cos\omega t) + \\ & + Y_{22}(\cos\omega t - \frac{a}{A}\sin\varphi\sin\omega t) \end{aligned} \qquad (14)$$

where we have considered:

$$Y_{11} = (A + a\cos\varphi)(N_R(A) + AN'_R(A))$$

$$Y_{22} = (A + a\cos\varphi)(N_I(A) + AN'_I(A))$$

under the condition that the describing function N(A) is differentiable with respect to *A*.

Processing equation (14) and neglecting the terms containing powers of $\frac{a}{A}$ greater than one, we obtain:

$$y(t) \cong y_1(t) + y_2(t) + y_3(t) \qquad (15)$$

where:

$$y_1(t) = A\left[N_R(A)\sin\omega t + N_I(A)\cos\omega t\right]$$

$$\begin{aligned} y_2(t) = & a\left[N_R(A)\sin(\omega t + \varphi)\right] + \\ & + a\left[N_I(A)\cos(\omega t + \varphi)\right] \end{aligned}$$

$$\begin{aligned} y_3(t) = & A\,a\cos\varphi\left[N'_R(A)\sin\omega t\right] + \\ & + A\,a\cos\varphi\left[N'_I(A)\cos\omega t\right] \end{aligned}$$

Let us consider the complex images of the nonlinear element's output signal components produced by the element's input signal's very small amplitude component:

$$\begin{aligned} Y_2(A, a, \omega, \varphi) = & a\left[N_R(A)\exp j(\omega t + \varphi)\right] + \\ & + a\left[N_I(A)\exp j(\omega t + \varphi + \frac{\pi}{2})\right] \end{aligned}$$

$$\begin{aligned} Y_3(A, a, \omega, \varphi) = & Aa\cos\varphi\left[N'_R(A)\exp j\omega t\right] + \\ & + Aa\cos\varphi\left[N'_I(A)\exp j(\omega t + \frac{\pi}{2})\right] \end{aligned}$$

The complex image of the small amplitude input signal of the nonlinear element is:

$$X_2(a,\omega,\varphi) = a\exp j(\omega t + \varphi)$$

The ratio of this complex functions is the incremental input describing function $N_i(A,\varphi)$:

$$N_i(A,\varphi) = \frac{Y_2(A,a,\omega,\varphi) + Y_3(A,a,\omega,\varphi)}{X_2(A,a,\omega,\varphi)} \quad (16)$$

and has the form:

$$N_i(A,\varphi) = N(A) + A\cos\varphi N'(A)\exp(-j\varphi) \quad (17)$$

Considering:

$$(\cos\varphi)\exp(-j\varphi) = (\cos\varphi)(\cos\varphi - j\sin\varphi) = \\ = \frac{1}{2}(1+\cos 2\varphi - j\sin 2\varphi)$$

or

$$(\cos\varphi)\exp(-j\varphi) = \frac{1}{2}(1+\exp(-j2\varphi))$$

equation (16) becomes:

$$N_i(A,\varphi) = \\ = N(A) + AN'(A)\frac{1+\exp(-j2\varphi)}{2} \quad (18)$$

The incremental input describing function is independent of the amplitude and the frequency of the very small amplitude signal, known as incremental input signal.

2. ANALYSIS OF INCREMENTAL SYSTEM

Defining the incremental input describing function allows assessing the stability of a system with the structure of Fig.1, with known initial conditions. Therefore it is possible to determine the stability or instability (in particular the occurrence of the jump resonance phenomenon) of the forced oscillation of frequency ω, perturbed by the same frequency but very small amplitude signal, a.
The equivalent incremental structure of the system in Fig.1 is presented in Fig.2.

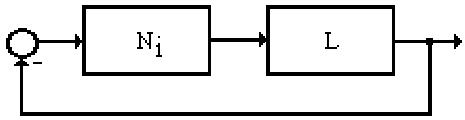

Figure 2: Equivalent of incremental structure.

The characteristic equation of the equivalent system in Fig.2 is:

$$1 + H(j\omega)N_i(A,\varphi) = 0 \quad (19)$$

or

$$H(j\omega) = -\frac{1}{N_i(A,\varphi)} \quad (20)$$

The inverse negative incremental describing locus is the curve plotted by the vector $-\frac{1}{N_i(A,\varphi)}$, similar to the inverse negative describing locus $-\frac{1}{N(A)}$.
For a constant value of amplitude A and a variation of 2π of the phase difference φ, the trajectory of the vector $-\frac{1}{N_i(A,\varphi)}$ is called stability curve and, in the polar plane, is a circle.
This geometrical interpretation is a direct consequence of equation (17) if we rewrite it in the following way:

$$N_{iR}(A,\varphi) + jN_{iI}(A,\varphi) = N(A) + \\ + \frac{AN'(A)}{2}(1+\cos 2\varphi) - \\ - j\frac{AN'(A)}{2}\sin 2\varphi \quad (21)$$

where $N_{iR}(A)$, $N_{iI}(A)$ denote the real and imaginary part of the incremental input describing function. Equation (21) yields:

$$\begin{cases} N_{iR}(A,\varphi) = N(A) + \frac{AN'(A)}{2}(1+\cos 2\varphi) \\ N_{iI}(A,\varphi) = -\frac{AN'(A)}{2}\sin 2\varphi \end{cases} \quad (22)$$

and eliminating the phase difference φ between the two equations (22) we obtain:

$$\left[N_{iR}(A,\varphi)-\left(N(A)+\frac{AN'(A)}{2}\right)\right]^2 + \\ + N_{iI}(A,\varphi)=\left[\frac{AN'(A)}{2}\right]^2 \qquad (23)$$

In the plane $((N_{iR}(A,\varphi), jN_{iI}(A,\varphi))$, equation (23) for a constant value of the amplitude A, actually represents the equation of a circle with the center in the point of coordinates $(N(A)+\frac{AN'(A)}{2},0)$ and of radius $(\frac{AN'(A)}{2})$, Fig.3.

In the plane $(-1/N_{iR}(A,\varphi),-j/N_{iI}(A,\varphi))$, the stability curve looks like in Fig.4.

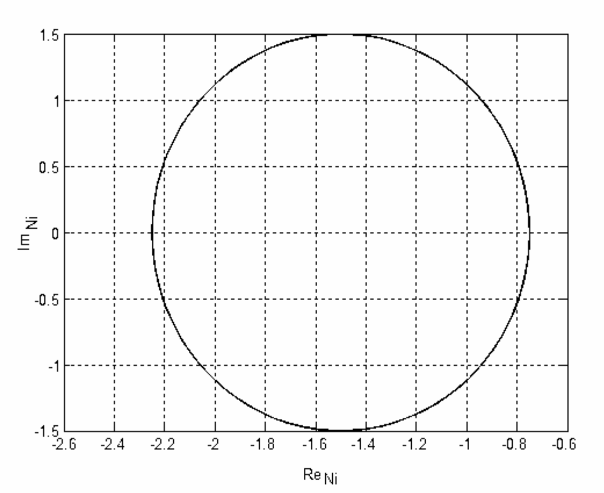

Figure 3: Complex representation of $N_i(A,\varphi)$ for *N(A)* real; *A*=1.

The stability or the instability of the forced oscillation of frequency ω according to equation (20) is geometrically determined, depending on the position of the transfer locus of the linear element $H(j\omega)$ relative to the curve plotted by the vector $-\frac{1}{N_i(A,\varphi)}$, for a constant value of the amplitude A and a 2π variation of the phase difference.

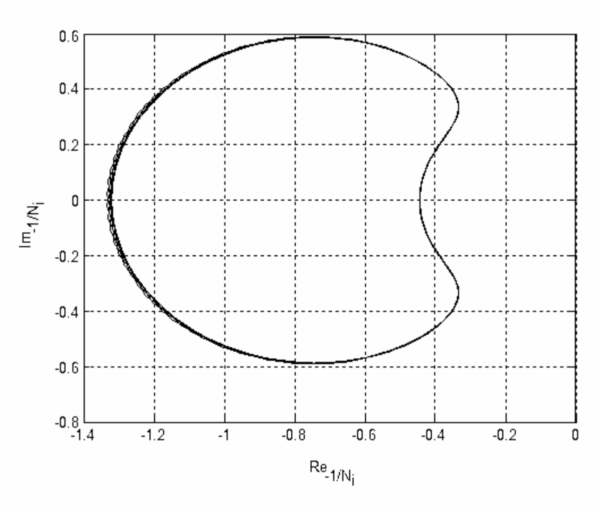

Figure 4: Complex representation of $-1/N_i(A,\varphi)$ for *N(A)* real; *A*=1.

3. CONCLUSIONS

We have shown the possibility of estimating the stability of a nonlinear automatic system using a graphical-analytical method. Considering that the system is perturbed and thus the nonlinearity's input signal has two harmonic components of same frequency but different amplitudes, we have described the behavior of the nonlinear element using the incremental input describing function. Solving the characteristic equation of the equivalent incremental system, through a graphical representation there is the possibility to estimate the stability of the system.

References

[1] A. Gelb, W.E. Vander Velde, *Multiple-Input Describing Functions and Nonlinear Systems Design*, McGraw-Hill Book Company, New-York, 1968.

[2] M. Vidyasagar, *Nonlinear Systems Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1993.

[3] M. Cleju, *On the Frequency Response of Nonlinear Systems*. Proc.of the International Symposium on Systems Theory, Craiova, Romania, 2000, pp. A-100-A-102.

[4] P.H. Vho, *Nonlinear Dynamic Responses of a Motor Torque Brush*, The Nonlinear Journal, vol. 1, 1999, pp. 29-40.