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ON THE STABILITY OF SYSTEMS WITH SATURATION AND MULTIPLE DELAY IN COMMAND AND MULTIPLE DELAY IN STATE

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***Abstract* – In this section we consider systems with multiple delay in state and command and saturation in command, and using a transformation given in [5], [6], the initial system is transformed in one whithout dealy but which contain saturation in command. The investigations are continuing using some results from the study of systems with saturation in command [2], [3]. In this manner, using the transformation relation between the state of the initial system with delay and the state of the transformed system without delay, we can formulate some results regarding the stabilization of the initial system with multiple delay and saturation in command and multiple delay in state.The Propositions 1..6 from this paper are personal results of the author.**

Keywords: multiple delay in command and state, stabilization, saturation in command

1. INTRODUCTION

A general method for transformation of systems with delay in command and state is presented in [5] and [6]. In those paper are demonstrate how many problems of stabilization, controllability can be dealt with by addressing the reduced(associate) systems. The reduction provides, therefore, a strong tool for manipulating systems with delays in state and control.

In [2] and [3] are presented some necessary and sufficient conditions for global asymptotic stability of linear systems with bounded control.

Starting from these, although in practice, control bounds and delayed are usually ignored in the initial design, the aim of this paper is to find under what conditions will the equilibrium of a system with multiple delay in state and command and saturation in command, remain globally asymptotically stable.

In this paper are presented results about stability, instability and a estimation of stability region for the considered systems. The Propositions 1..6 from this paper are personal results of the author. Similar results about systems with delay in command and saturation in command, systems with multiple delay in command and saturation in command, systems with delay in state and command and saturation in command and systems with distributed delay in state and command and saturation in command, are presented by author in [7].

2. MAIN RESULTS

We consider the monovariabil system in the following form :

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-h_i) + B_0 u_s(t) + \sum_{i=1}^{k} B_i u_s(t-h_i) \quad (1)$$

where $x \in \Re^n$ is the state, h_i, $i = 1,\dots,k$ are the delays in command and state(we consider same delays in state and comand just for a easy redactation), A_0, A_i, B_0, B_i are matrices of appropriate dimensions. The initial conditions of command are given by a function $u_{s0}(\cdot)$ definited on the interval $[-h,0]$, and the initial condition of state are given by a function $x_0(\cdot)$ definited on the interval $[-h,0]$, where $h = \max\{h_1,\dots,h_k\}$. The command contain saturation and is in the general form :

$$u_s(t) = -sat(Kx(t)) = -\mu(x(t))Kx(t) \quad (2)$$

$$\text{where } \mu(x) = \begin{cases} 1 & if \quad |Kx| < u_{\lim} \\ \dfrac{u_{\lim}}{|Kx|} & if \quad |Kx| \geq u_{\lim} \end{cases} \quad (3)$$

$u_{\lim}$ is the maxim value of command, $|u_s| \leq u_{\lim}$, K is a feedback matrix.

Let the system (1), and use the state transformation :

$$y(t) = x(t) + \sum_{i=1}^{k} \int_{t-h_i}^{t} e^{(t-s-h_i)A} A_i x(s) ds + \sum_{i=1}^{k} \int_{t-h_i}^{t} e^{(t-s-h_i)A} B_i u_s(s) ds \quad (4)$$

where A is a solution of the matrix equation (we suppouse that exist a solution) :

$$A = A_0 + \sum_{i=1}^{k} e^{-Ah_i} A_i \quad (5)$$

We note : $s = t + \theta$, and comupting $\dot{y}$, we obtain:

$$\dot{y}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-h_i) + B_0 u_s(t) + \sum_{i=1}^{k} B_i u_s(t-h_i) +$$

$$+ \sum_{i=1}^{k} \int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i \dot{x}(t+\theta) d\theta + \sum_{i=1}^{k} \int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i \dot{u}_s(t+\theta) d\theta$$

$$\int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i \dot{x}(t+\theta)d\theta = e^{-A(\theta+h_i)} A_i x(t+\theta)\,|_{-h_i}^{0} +$$

$$+\int_{-h_i}^{0} Ae^{-A(\theta+h_i)} A_i x(t+\theta)d\theta =$$

$$= e^{-Ah_i} A_i x(t) - A_i x(t-h_i) + \int_{-h_i}^{0} Ae^{-A(\theta+h_i)} A_i x(t+\theta)d\theta$$

$$\int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i \dot{u}_s(t+\theta)d\theta = e^{-A(\theta+h_i)} B_i u_s(t+\theta)\,|_{-h_i}^{0} +$$

$$+\int_{-h_i}^{0} Ae^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta =$$

$$= e^{-Ah_i} B_i u_s(t) - B_i u_s(t-h_i) + \int_{-h_i}^{0} Ae^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta$$

Observing that the sum formed by the last integral of each up terms for $i=1,...,k$ is equal $A(y(t)-x(t))$, using (5) and making the replacement up, we obtain the associate system :

$$\dot{y}(t) = Ay(t) + (B_0 + \sum_{i=1}^{k} e^{-h_i A} B_i)u_s(t) \qquad (6)$$

We make the notation :

$$B = B_0 + \sum_{i=1}^{k} e^{-h_i A} B_i \qquad (7)$$

We suppose that the comand of (1) contain saturation and is in the form :

$$u_s(t) = -\mu(x(t) + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta)\cdot$$

$$\cdot K\left[x(t) + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta\right] \qquad (8)$$

where the matrix A is given by (5) and :

$$\mu(x + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta) =$$

$$= 1 \qquad if$$

$$\left|K(x + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta)\right| < u_{\lim}$$

$$= \frac{u_{\lim}}{\left|K(x + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta)\right|}$$

$$if \qquad (9)$$

$$\left|K(x + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta)\right| \geq u_{\lim}$$

$u_{\lim}$ is the maxim value of command, $|u_s| \leq u_{\lim}$, K is a feedback matrix. We reconsider the monovariable associate system (6) :

$$\dot{y}(t) = Ay(t) + Bu_s(t), \qquad (10)$$

where $y \in \Re^n$ is the state, A, B are matrices of appropriate dimensions. The command of this system contain saturation and is in the form :

$$u_s(t) = -sat(Ky) = -\mu(y(t))Ky(t), \qquad (11)$$

$$\text{where } \mu(y) = \begin{cases} 1 & if \quad |Ky| < u_{\lim} \\ \dfrac{u_{\lim}}{|Ky|} & if \quad |Ky| \geq u_{\lim} \end{cases} \qquad (12)$$

$u_{\lim}$ is the maxim value of command, $|u_s| \leq u_{\lim}$, K is a feedback matrix.

Observation 1 : In [5], [6] is claimed that : if $u(t) = F(\cdot)y(t)$ is a stabilyzing law for system (10), then the next command law :

$$u(t) = F(\cdot)\left[x(t) + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \sum_{i=1}^{k}\int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta\right] \qquad (13)$$

is stabilyzing for the system (1), under the condition that all unstable eigenvalues of system (1) are contained in the spectrum of the matrix A given by (5).

Definition 1 : Let $A_i \in \Re^{nxn}$. A set $\{A_1,...,A_k\}$ is *simultaneously P Liapunov stable* , if there exists a $P \in \Re^{nxn}$, pozitive definite, such that $A_i^T P + PA_i < 0,\ i=1,...,k$. [2]

With these we claim :

Proposition 1 : The null solution of closed loop system (1), (8) and (9) is globally asimptotically stable if there exist K and $P \in \Re^{nxn}$ pozitive definite, such that the set $\left\{A, A-(B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K\right\}$ is *simultaneously P Liapunov stable*, namely : $A^T P + PA < 0$ and

$$(A - (B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K)^T P + P(A - (B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K) < 0$$

under the condition that all unstable eigenvalues of system (1) are contained in the spectrum of the matrix A given by (5).

Proof : We use a result from [2], given by

Theorem 1 : The null solution of closed loop system (10), (11) and (12) is globally asimptotically stable if there exist K and $P \in \Re^{nxn}$, pozitive definite, such that the set $\{A, A-BK\}$ is *simultaneously P Liapunov stable*, namely :

$A^T P + PA < 0$ and $(A-BK)^T P + P(A-BK) < 0$

Proof of Theorem 1 **:** Let consider the Lyapunov function : $V(y) = y^T Py$, and the matrix $P > 0$ who satisfy the hypothesis. With these we obtain :

$$y^T (A^T P + PA)y = -y^T Qy < 0 \text{ and}$$

$$y^T ((A-BK)^T P + P(A-BK))y = -y^T Qy + y^T My < 0$$

where $Q > 0$ and $M = -(PBK + K^T B^T P)$.

Then one obtains $y^T My < y^T Qy$. As $\mu(y) \in (0,1]$ it follows that :

$$\dot{V}(y) = -y^T Qy + \mu(y) y^T My < -y^T Qy + \mu(y) y^T Qy \leq$$

$\leq -y^T Qy + y^T Qy = 0$, and the proof of Theorem 1 is finished. □

Applying the Theorem 1 on the system (10), where A and B are given by (5) and (7) respectively, using the Observation 1 where $F(\cdot) = -\mu(y)K$ and y is given by (4), then the proof of Proposition 1 is finished. □

Definition 2 : Two diagonalizable matrices $A, B \in \Re^{nxn}$, are said to be *simultaneously diagonalizable* if there exists a single non-singular matrix N such that $N^{-1}AN$ and $N^{-1}BN$ are both diagonal. [2]

Lema 1 : Let A and B be diagonalizable from $\Re^{nxn}$. Then A and B are *simultaneously diagonalizable* if and only if A and B commute under multiplication, namely $AB = BA$. [2]

Proposition 2 : The null solution of closed loop system (1), (8) and (9) is globally asimptotically stable under the condition that all unstable eigenvalues of system (1) are contained in the spectrum of the matrix A given by (5) and if are true:

a) the matrix A is exponentially stable and diagonalizable

b) the matrix $A - (B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K$ is exponentially stable and diagonalizable

c) the matrices A and $(B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K$ commute under multiplication

Proof : We use a result from [2], given by

Theorem 2 : The null solution of closed loop system (10), (11) and (12) is globally asimptotically stable if are true :

a) the open-loop system A is exponentially stable and diagonalizable

b) the matrix $A - BK$ is exponentially stable and diagonalizable

c) the matrices A and BK commute under multiplication

Proof of Theorem 2 : Since A and BK commute, then A and $A-BK$ commute. By assumption, A and $A-BK$ are also diagonalizable. Thus from Lema 1.1, A and $A-BK$ are simultaneously diagonalizable. Thus, there exists a coordinate transformation T such that A and $A-BK$ are both diagonal with respect to a new coordinate $z = Ty$. Let $\overline{A} = A - BK$ and let $\Lambda_A, \Lambda_{\overline{A}}$ be diagonal matrices where : $\Lambda_A = TAT^{-1}$, $\Lambda_{\overline{A}} = T(A-BK)T^{-1}$. Then we proof that $P = T^T T$ satisfies the conditions of Theorem 1.

$2\Lambda_A = TAT^{-1} + (TAT^{-1})^T$, and multiplying the left side by T^T and the right side by T, we obtain :

$2T^T \Lambda_A T = T^T (TAT^{-1} + (TAT^{-1})^T)T = T^T TA + A^T T^T T =$

$= PA + A^T P$ where $P = T^T T > 0$ and $T^T \Lambda_A T < 0$ since T is non-singular.

Similary : $2T^T \Lambda_{\overline{A}} T = T^T T\overline{A} + \overline{A}^T T^T T$. Thus, P simultaneously satisfies $A^T P + PA < 0$ and $\overline{A}^T P + P\overline{A} < 0$. By Theorem 1, the proof of Theorem 2 is finished. □

Applying the Theorem 2 on the system (10), where A and B are given by (5) and (7) respectively, using the Observation 1 where $F(\cdot) = -\mu(y)K$ and y is given by (4), then the proof of Proposition 2 is finished. □

A analog result is given by :

Proposition 3 : The null solution of closed loop system (1), (8) and (9) is globally asimptotically stable under the condition that all unstable eigenvalues of system (1) are contained in the spectrum of the matrix A given by (5) and if are true:

a) A and $A - (B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K$ are exponentaially stable

b) $A - (B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K$ is diagonalizable

c) $\hat{A}$ commutes with P, where $\hat{A}$ is the diagonal form of $A - (B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K$, and $P > 0$ solves : $A^T P + PA < 0$.

Proof : We use a result from [2], given by

Theorem 3 : The null solution of closed loop system (10), (11) and (12) is globally asimptotically stable if are true :

a) A and $A - BK$ are exponentaially stable

b) $A - BK$ is diagonalizable

c) $\hat{A}$ commutes with P, where $\hat{A}$ is the diagonal form of $A - BK$, and $P > 0$ solves : $A^T P + PA < 0$.

Proof of Theorem 3 : Let $\hat{A} = T(A-BK)T^{-1}$ where

T diagonalizes $A-BK$ and $\widehat{A}$ is diagonal in the new coordinate $z=Tx$.
Also let $\overline{A}=TAT^{-1}$. Since $\overline{A}$ is exponentially stable, there exists $P>0$ such that $\overline{A}^TP+P\overline{A}<0$.
Since $\widehat{A}<0$ and $P>0$, all eigenvalues of $\widehat{A}P$ are less than zero. Also, by assumption, $\widehat{A}P=P\widehat{A}$, $\widehat{A}^T=\widehat{A}$ wich implies that $\widehat{A}^TP+P\widehat{A}<0$. By Theorem 1, the proof of Theorem 3 is finished. □
Applying the Theorem 3 on the system (10), where A and B are given by (5) and (7) respectively, using the Observation 1 where $F(\cdot)=-\mu(y)K$ and y is given by (4), then the proof of Proposition 3 is finished. □
For multivariable systems we present the next result :
Proposition 4 : We consider the system (1) in the multivariable form :

$$\dot{x}(t)=A_0x(t)+\sum_{i=1}^{k}A_ix(t-h_i)+B_0u_s(t)+\sum_{i=1}^{k}B_iu_s(t-h_i) \quad (14)$$

where $x\in\Re^n$ is the state, h_i, $i=1,...,k$ are the delays in command and state, A_0, A_i, B_0, B_i are matrices of appropriate dimensions, $u\in\Re^m$, under the condition that all unstable eigenvalues of system (14) are contained in the spectrum of the matrix A given by (5). We note B_j^* the jth column of $B_0+\sum_{i=1}^{k}e^{-Ah_i}B_i$ and we assume that A is asymptotically stable. The inputs are $u_s=[u_{s1}, \;\dots\; ,u_{sm}]^T$, $u_{\max j}$ is the maxim value of the component jth of command namely $|u_{sj}|<u_{\max j}, j=1,...,m$. The initial conditions of commands are given by a set of functions $u_{s0j}(\cdot)$ definited on the interval $[-h,0]$ and bounded by $u_{\max j}$. The initial conditions of state are given by a functions $x_0(\cdot)$ definited on the interval $[-h,0]$, where $h=\max\{h_1,...,h_k\}$. The components of command are in the form : $u_{sj}=$ (15)

$$=-B_j^{*T}P(x+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}A_ix(t+\theta)d\theta+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}B_iu_s(t+\theta)d\theta) \;\; if$$

$$\left|B_j^{*T}P(x+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}A_ix(t+\theta)d\theta+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}B_iu_s(t+\theta)d\theta)\right|<u_{\max j}$$

$$=-\mu_jB_j^{*T}P(x+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}A_ix(t+\theta)d\theta+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}B_iu_s(t+\theta)d\theta) \;\; if$$

$$\left|B_j^{*T}P(x+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}A_ix(t+\theta)d\theta+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}B_iu_s(t+\theta)d\theta)\right|\geq u_{\max j}$$

where

$$\mu_j=\frac{u_{\max j}}{\left|B_j^{*T}P(x+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}A_ix(t+\theta)d\theta+\sum_{i=1}^{k}\int_{-h_i}^{0}e^{-A(\theta+h_i)}B_iu_s(t+\theta)d\theta)\right|}$$

$$j=1,...,m \quad (16)$$

If $P>0$ solves $A^TP+PA<0$, then the null solution of closed loop system (14), (15) and (16) is globally asimptotically stable.
Proof : We use a result from [2], given by
Theorem 4. We consider the multivariable system in the following form:

$$\dot{y}=Ay+Bu_s=Ay+\sum_{i=1}^{m}B_iu_{si}\,, \quad (17)$$

where : $y\in\Re^n, u_s\in\Re^m, A\in\Re^{nxn}$ is asymptotically stable, $B\in\Re^{nxm}$, B_i is the ith column of B. The inputs are $u_s=[u_{s1}, \;\dots\; ,u_{sm}]^T$, $u_{\max i}$ is the maxim value of the component ith of command namely $|u_{si}|<u_{\max i}, i=1,...,m$.
The command vector $u_s=-sat(B^TPy)$, have the components in form :

$$u_{si}=\begin{cases}-B_i^TPy\;; & |B_i^TPy|<u_{\max i}\\ -\mu_iB_i^TPy\;; & |B_i^TPy|\geq u_{\max i}\end{cases}, \quad (18)$$

$$\text{where } \mu_i=\frac{u_{\max i}}{|B_i^TPy|}, \qquad i=1,...,m\,, \quad (19)$$

If $P>0$ solves $A^TP+PA<0$, then the null solution of closed loop system (17), (18) and (19) is globally asimptotically stable.
Proof of Theorem 4 **:** We can rewrite the command vector : $u_s=-MB^TPy$, where :
$M=diag(\beta_i), M\in\Re^{mxm}, \beta_i\in(0,1]$ and

$$\beta_i=\begin{cases}1 & if \quad |B_i^TPy|<u_{\max i}\\ \mu_i & if \quad |B_i^TPy|\geq u_{\max i}\end{cases}$$

Let consider the Lyapunov function : $V(y)=y^TPy$ and computing $\dot{V}(y)$, we obtain :
$\dot{V}(y)=y^T[(A-BMB^TP)^TP+P(A-BMB^TP)]y=$
$=y^T(A^TP+PA-2PBMB^TP)y<0$,since
$PBMB^TP\geq 0$ and $A^TP+PA<0$.

Thus the proof of Theorem 4 is finished. □

Applying the Theorem 4 on the system (10) considered now multivariable, where A and B are given by (5) and (7) respectively, using the Observation 1 where $F(\cdot) = -MB^T P$ and y is given by (4), then the proof of Proposition 4 is finished. □

The next two propositions are concerning on the open loop unstable monovariable linear systems.

Proposition 5 : We consider the system (1) and A is given by (5). Suppose A is invertible and has a single unstable eigenvalue λ.

Let $x_{eq} = \pm A^{-1}(B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)u_{\lim}$ denote the equilibrium points of the saturated system when the input saturates at $u_s = -u_{\lim}$ and $u_s = u_{\lim}$ respectively. Then, no feedback matrix K where $\left|Kx_{eq}\right| \geq u_{\lim}$, can globally stabilize the null solution of closed loop system (1), (8) and (9).

Proof : We use a result from [2], given by

Theorem 5 : We consider the system (10) and suppose A is invertible and has a single unstable eigenvalue λ. Let $y_{eq} = \pm A^{-1}Bu_{\lim}$ denote the equilibrium points of the saturated system when the input saturates at $u_s = -u_{\lim}$ and $u_s = u_{\lim}$ respectively. Then, no feedback matrix K where $\left|Ky_{eq}\right| \geq u_{\lim}$, can globally stabilize the null solution of closed loop system (10), (11) and (12).

Proof of Theorem 5 : To show that the origin is not globally asymptotically stable, it is sufficient to find some initial conditions $y_0 \in \Re^n$ wich cannot be driven to the origin with the feedback :
$u_s(t) = -sat(Ky) = -\mu(y(t))Ky(t)$ where K satisfy $\left|Ky_{eq}\right| \geq u_{\lim}$. Let $E_\lambda(y_{eq})$ be the eigenspace corresponding to the unstable eigenvalue λ of the open-loop system A where :

$$E_\lambda(y_{eq}) = \left\{ y \in \Re^n : A(y - y_{eq}) = \lambda(y - y_{eq}) \right\} \quad (20)$$

We will show that some initial conditions on the eigenspace E_λ cannot be driven to the origin with the feedback $u_s(t) = -sat(Ky)$.

Note that $|Ky| = u_{\lim}$ depicts the saturation boundaries. Now consider the case when saturation occurs with $u_s = -u_{\lim}$. Then, the dynamics of the saturated system are given by :

$$\dot{y}(t) = Ay(t) - Bu_{\lim}, \quad (21)$$

and the equilibrium point under saturation by :

$$y_{eq} = A^{-1}Bu_{\lim} \quad (22)$$

Let $D = \left\{ y : |Ky| \geq u_{\lim} \right\}$. The assumption $\left|Ky_{eq}\right| \geq u_{\lim}$ implies $y_{eq} \in D$. Then the trajectory $y(t)$ for the saturated system when $y_0 \in E_\lambda(y_{eq})$ is given by :

$$y(t) = e^{\lambda t}(y_0 - y_{eq}) + y_{eq}, \quad (23)$$

Moreover, since $E_\lambda(y_{eq})$ is the eigenspace, $y(t) \in E_\lambda(y_{eq}) \cap D$ provided the system remains saturated at $u_s = -u_{\lim}$. We will show that some initial conditions $y_0 \in E_\lambda \cap D$ exist where $y(t)$ never leaves the saturated region D so that $|y(t)|$ becomes unbounded.

Now, E_λ is either parallel to or intersects $Ky = u_{\lim}$. Because $Ky = u_{\lim}$ forms an $n-1$ dimensional surface and $E_\lambda(y_{eq})$ a line, the intersection is a point. Suppose $E_\lambda(y_{eq})$ and $Ky = u_{\lim}$ are parallel. Since $y_{eq} \in D$, $E_\lambda(y_{eq})$ lies entirely in the saturated region. This means that :

$\forall y_0 \in E_\lambda(y_{eq}),\ y(t) \in E_\lambda(y_{eq}),\ \forall t \geq 0$. Since $E_\lambda(y_{eq})$ is an unstable eigenspace, $|y(t)|$ will become unbounded.

Now suppose $E_\lambda(y_{eq})$ and $Ky = u_{\lim}$ intersect. Let v^* denote the point of intersection. Then $\forall y_0 \in E_\lambda(y_{eq}) \cap D$ such that $|y_0| \geq \max\left(\left|v^*\right|, \left|y_{eq}\right|\right)$, $y(t) \in E_\lambda(y_{eq}) \cap D,\ t \geq 0$ and $|y(t)|$ will become unbounded.

The same argument can be repeated for saturation occurring at $u_s = u_{\lim}$. Thus, there exist initial conditions on the eigenspace corresponding to the unstable eigenvalue which becomes unbounded. Hence, the origin is not globally asymptotically stable under any linear time invariant state feedback. Thus the proof of Theorem 5 is finished. □

Applying the Theorem 5 on the system (10), where A and B are given by (5) and (7) respectively, and using the transformation relation given by (4), then the proof of Proposition 5 is finished. □

The next proposition examines the region of stability for an open loop unstable system under control constraints and delay in state and control.

Proposition 6 : We consider the system (1) where A is given by (5). Suppose the following are true :

a) matrix A is unstable.

b) matrix $A - (B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K$ is exponentially stable. Let :

$$B_d^* = \left\{ x : \left(x(t) + \sum_{i=1}^{k} \int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \sum_{i=1}^{k} \int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta \right)^T P \cdot \right.$$

$$\left. \cdot \left(x(t) + \sum_{i=1}^{k} \int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \sum_{i=1}^{k} \int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta \right) \le d \right\}$$

, $d \in \Re_+$ and

$$H^* = \left\{ x : \left| K(x(t) + \sum_{i=1}^{k} \int_{-h_i}^{0} e^{-A(\theta+h_i)} A_i x(t+\theta)d\theta + \right. \right.$$

$$\left. \left. + \sum_{i=1}^{k} \int_{-h_i}^{0} e^{-A(\theta+h_i)} B_i u_s(t+\theta)d\theta) \right| \le u_{\lim} \right\},$$ where $P > 0$ is a solution to :

$$(A - (B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K)^T P + P(A - (B_0 + \sum_{i=1}^{k} e^{-Ah_i} B_i)K) < 0$$

Then $B_{d^*}^*$ is an exponentially stable region for the closed loop system (1), (8) and (9), where d^* is the largest number such that $B_{d^*}^* \subset H^*$.

Proof : We use a result from [2], given by

Theorem 6 : We consider the system (10) and suppose the following are true :

a) matrix A is unstable.

b) matrix $A - BK$ is exponentially stable.

Let $B_d = \{ y : y^T P y \le d \}$, $d \in \Re_+$ and $H = \{ y : |Ky| \le u_{\lim} \}$, where $P > 0$ is a solution to $(A - BK)^T P + P(A - BK) < 0$. Then B_{d^*} is an exponentially stable region for the closed loop system (54), (55) and (56), where d^* is the largest number such that $B_{d^*} \subset H$.

Proof of Theorem 6 : Since $A - BK$ is exponentially stable, there exist $P > 0$, such that : $(A - BK)^T P + P(A - BK) < 0$. Let consider the Lyapunov function : $V(y) = y^T P y$, and computing $\dot{V}(y)$ we obtain :

$$\dot{V}(y) = y^T [(A - BK)^T P + P(A - BK)] y < 0 .$$

In addition, B_{d^*} is the largest set wich lies within the unsaturated region H. Thus $\forall y \in B_{d^*}$, $y^T P y$ decreases and hence $|y| \to 0$ exponentially. Thus the proof of Theorem 6 is finished.

Applying the Theorem 6 on the system (10), where A and B are given by (5) and (7) respectively, and using the transformation relation given by (4), then the proof of Proposition 6 is finished.

3. CONCLUSIONS

In this paper we consider systems with multiple delay in state and command and saturation in command, and using a transformation given in [5], the initial system is transformed in one whithout dealy but which contain saturation in command. The investigations are continuing using some results from the study of systems with saturation in command [2], [3]. In this manner, using the transformation relation between the state of the initial system with delay and the state of the transformed system without delay, we can formulate some results regarding the stabilization of the initial system with multiple delay in state and command and saturation in command.

Are presented results about stability, instability and a estimation of stability region for the considered systems. The Propositions 1..6 from this paper are personal results of the author. Similar results about systems with delay in command and saturation in command, systems with multiple delay in command and saturation in command, systems with delay in state and command and saturation in command and systems with distributed delay in state and command and saturation in command, are presented by author in [7].

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