6TH INTERNATIONAL CONFERENCE ON ELECTROMECHANICAL AND POWER SYSTEMS
October 4-6, 2007 - Chişinău, Rep.Moldova

STEADY STATE MAGNETIC FIELD COMPUTATION USING THE SYMBOLIC METHODS

Gheorghe GAVRILĂ, Florin ENACHE, Emil CAZACU

Military Technical Academy, gavrila_g@mta.ro, florinenache@mta.ro,
"Politehnica" University of Bucharest, cazacu_emil@yahoo.com

Abstract **– In the case of the designing of the different types of the electromagnetic devices, a particular important problem is the magnetic field computation in different operation. The problem is itself a difficult one, even in the steady-state and/or quasi steady-state. After a series of considerations are done with regard to the magnetic field computation methods, emphasizing the fact that the things are more complicated than in the case of the electric field computation, in the paper are presented the steady-state magnetic field. Then the results of the magnetic field computation are given back in the case of three applications: the straight filamentary conductor with the finite length, the straight circular single turn and the bar with the rectangular cross-section. The study is accomplished by means of the symbolic computation (MAPLE) which presents a series of advantages in comparison with the computation on the basis of the numerical methods (FEM, FDM, etc.).**

Keywords: *steady-state magnetic field, vector magnetic potential, symbolic methods.*

1. INTRODUCTION

The solving of an electromagnetic field problem generally, respectively of a steady-state magnetic field problem, particularly, requires the crossing of three stages, namely: - the correct formulation of the field problem; - the choice of the most suitable method for the solving; - the verification of the results by means of the particular solutions or by their comparison with the others, which are a priori known and was obtained by others methods than the chosen one.

The correct formulation of a field problem supposes, first the establishment of a phenomenological model of this. The transposition in the formal plan of the essential phenomena, on the base of a biunique correspondence, constitutes what is named mathematic model. The phenomenological model and the mathematic one form the field theoretic model. The afferent equations of the field theoretic model have to form a complete system that is to satisfy the existence and uniqueness theorems.

The field problem is correct formulated if the solution exists, is unique and depends continuous of the problem data.

In the main, field mathematic model can be: of the differential, integral, variational or topological type. The solution of the field problem can be obtained by: *analytical, numerical and graphical methods*.

The main analytical methods are: *the direct method (or the integral equations method), the magnetic image method, the method of the solving Laplace or Poisson equations by: the separation variables method, Green functions method, the complex variable functions method, the conformal transformations method (Schwarz-Christoffel),* etc.

The numerical methods can be applied, generally, for any field problem, two-dimensional (2D) or three-dimensional (3D). In this category enter: *finite difference method* (FDM), *boundary element method* (BEM), *finite element method* (FEM) *Monte-Carlo method* (MCM), etc.

The graphical methods consist in the graphic drawing of the field and equipotential lines spectra (Lehmann), and the graphic-analytical methods consist in the approximation of the field lines shape by the straight line segments and the circle arcs, connected between them.

The computation methods of the steady-state magnetic field are in a large measure similarly with the electrostatic field methods [1], [2]. The difference between the two classes of the problems consist in the types of Poisson and Laplace equations which are satisfied by them: only scalar equations for the electrostatic field V, scalar equations for the scalar magnetic potential V_m and vectorial equations for the vector magnetic potential $\bar{A}$ [3].

The computation of the magnetic field strength $\bar{H}$ and the magnetic flux density $\bar{B}$ assume known: the geometrical configuration of the conductors and the material properties of the media – given by magnetization curves $B = B(H)$ for the nonlinear media, respectively magnetic permeability μ for the linear media, as well as the currents intensities in the conductors and the total magnetic fluxes.

In both problems the boundary conditions are supposed that are known (*Dirichlet, Neumann or mixed*).

A special attention have to give to the boundary conditions for the vector magnetic potential in Dirichlet and Neumann problems, which are not

established as in the problem corresponding to the scalar potential, even in the cartesian co-ordinates.
In this paper they are presented three applications referring to steady-state magnetic field computation: the straight filamentary conductor with the finite length, the straight circular single turn and the bar with the rectangular cross-section. It is used the direct method, utilizing the Biot-Savart-Laplace formula in the homogeneous media on the basis of the facilities offered by the symbolic manipulator MAPLE [4], [5].

2. STEADY STATE MAGNETIC FIELD EQUATIONS

The magnetic field is represented by the vectors pair $\left(\overline{B}, \overline{H}\right)$. Fundamental relations of the steady-state magnetic field result from the general and material laws of electromagnetism in the following conditions: the bodies are motionless, $\overline{v} = 0$; the quantities are not variable in time, $\partial/\partial t = 0$. These relations are:

- *the coupling law between* $\overline{B}$, $\overline{H}$ *and* $\overline{M}$*:*

$$\overline{B} = \mu_0\left(\overline{H} + \overline{M}\right), \text{ with } \overline{M} = \overline{M_t} + \overline{M_p}; \tag{1}$$

- *the temporary magnetization:*

$$\overline{M_t} = f(\overline{H}), \text{ or } \overline{M_t} = \chi_m \overline{H} \quad \text{(isotropic materials);} \tag{2}$$

- *the magnetic flux law:*

$$\Phi_\Sigma = \oiint_\Sigma \overline{B} \cdot \overline{n}\, \mathrm{d}A = 0; \quad \mathrm{div}\, \overline{B} = 0; \tag{3}$$

- *the conservation theorem of the normal components of the magnetic flux density:*

$$\mathrm{div}_\mathrm{S} \overline{B} = \overline{n}_{12} \cdot (\overline{B_2} - \overline{B_1}) = 0; \; \left(B_n\right)_1 = \left(B_n\right)_2; \tag{4}$$

- *Ampère's law:*

$$u_{m\Gamma} = \oint_\Gamma \overline{H} \cdot \mathrm{d}\overline{l} = \Theta_{S_\Gamma}; \quad \mathrm{rot}\, \overline{H} = \overline{J}; \tag{5}$$

- *the conservation theorem of the tangential components of the magnetic field strength:*

$$\mathrm{rot}_\mathrm{S} \overline{H} = \overline{n}_{12} \times (\overline{H_2} - \overline{H_1}) = 0; \; \left(H_t\right)_1 = \left(H_t\right)_2 \tag{6}$$

or:

$$\mathrm{rot}_\mathrm{S} \overline{H} = \overline{n}_{12} \times (\overline{H_2} - \overline{H_1}) = \overline{J_l}; \left(H_t\right)_1 - \left(H_t\right)_2 = J_l, \tag{6'}$$

when on the discontinuity surface there is a superficial sheet of current;

- *the refraction theorem of the magnetic field lines:*

$$\frac{\mathrm{tg}\alpha_1}{\mathrm{tg}\alpha_2} = \frac{\mu_1}{\mu_2}; \tag{7}$$

- *the Poisson and Laplace equations:*

$$\Delta\overline{A} = -\mu\overline{J}, \quad \Delta\overline{A} = 0, \tag{8}$$

where $\overline{A}$ is vector magnetic potential.
From (3), $\mathrm{div}\, \overline{B} = 0$ it results:

$$\mathrm{rot}\, \overline{A} = \overline{B}, \tag{9}$$

with Coulomb gauge condition $\mathrm{div}\, \overline{A} = 0$

- *Biot-Savart-Laplace formula for the filiform conductors:*

$$\overline{H}\left(\overline{r}\right) = \frac{i}{4\pi} \oint_\Gamma \frac{\mathrm{d}\overline{s}' \times \overline{R}}{R^3}, \tag{10}$$

where $\left|\overline{R}\right| = \left|\overline{r} - \overline{r}'\right|$ is the distance from the elementary source of field to the point where the field is calculated.

- *Biot-Savart-Laplace formula for the solid conductors:*

$$\overline{H}\left(\overline{r}\right) = \frac{1}{4\pi} \iiint_{\mathfrak{D}} \frac{\overline{J}\left(\overline{r}'\right) \times \overline{R}}{R^3}\, \mathrm{d}v'. \tag{11}$$

In fact the solving of a steady-state magnetic field problem is reduced to the determination of the solution $\overline{A}(M) = \overline{A}(\overline{r})$ for the Poisson or Laplace equations, taking into account the boundary conditions. The verification of the solution is done by the comparison with a simple solution which is a priori known. Therefore, the solving of the simple problems is useful, because their solution constitutes the verifications for the results obtained in the case of the using of the modern methods for the steady-state magnetic field computation.
In many situations is more suitable that the vector magnetic potential $\overline{A}$ to be calculated firstly, and the magnetic flux density to be calculated with the relation (9). In the case of magnetic field generated by the filiform conductors which are crossed by the electric current, Biot-Savart-Laplace formula has the expression:

$$\overline{A}\left(\overline{r}\right) = \frac{\mu\, i}{4\pi} \oint_\Gamma \frac{\overline{\mathrm{d}s'}}{R}. \tag{12}$$

3. SYMBOLIC COMPUTATION. APPLICATIONS

Now there is the tendency to be used numerical methods for the electromagnetic field computation (FEM, FDM, etc.). However, the numerical approach in the electromagnetic field analysis has a series of the disadvantages: a) the study of the limit cases or of the result dependence of the problem parameters is done more difficult with the numerical methods, b) the using of the numerical methods leads often to the

loss of the physical meanings of the problem. These drawbacks can be eliminated by the using of symbolic methods, besides the numerical ones.
The main advantages of the utilization of the symbolic computations are:
- the automatic writing of the general expressions (in any point from the space) of the magnetic field (or of the vector magnetic potential) by the adequate choice of the co-ordinates system (function of the problem symmetry) and the accurate calculation of these;
- the automatic drawing of the 2D and 3D magnetic field spectra, allowing that the suggestive images to be obtained;
- the calculation of the particular solutions for which are known the simple formulas for the increasing of the confidence that the analysis was realized correct.
Using those which are showed in § 2, further the obtained results for three application are presented.

3.1. The magnetic field and the vector magnetic potential generated by the straight filamentary conductor with the finite length crossed by the electric current (fig. 1)

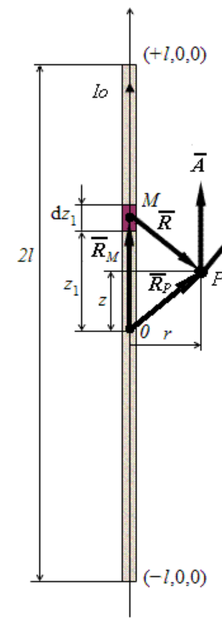

Figure 1: The straight filamentary conductor with the finite length crossed by the electric current.

In the table 1 is given the MAPLE code used for the symbolic computation of the vector magnetic potential and the magnetic field for the straight filamentary conductor with the finite length crossed by the electric current.

```
> e[r]:=vector([1,0,0]);e[phi]:=vector(
[0,1,0]);k:=vector([0,0,1]);
> Rp:=r*e[r]+z*k;Rm:=z1*k;Rp:=evalm(Rp);
Rm:=evalm(Rm);
>R:=Rp-Rm; R:=evalm(R);
>Rmod:=sqrt(R[1]^2+R[2]^2+R[3]^2);
> A:=mu*Io/(4*Pi)*Int(k/Rmod,z1=-l..l);
```

$$A_r = 0$$
$$A_\varphi = 0$$
$$A_z = \frac{\mu I_o}{4\pi} \ln \frac{\sqrt{(z-l)^2 + r^2} - (z-l)}{\sqrt{(z+l)^2 + r^2} - (z+l)}$$

```
>v:=[r,phi,z]:B:=curl([Ar,Aphi,Az],v,coo
rds=cylindrical):Br:=dotprod(B,e[r]);Bz:
=dotprod(B,k);Bphi:=dotprod(B,e[phi]);
```

$$B_r = 0$$
$$B_z = 0$$
$$B_\varphi = \frac{\mu I_o r}{4\pi} \frac{1}{\sqrt{(z-l)^2 + r^2}\sqrt{(z+l)^2 + r^2}} \cdot$$
$$\cdot \frac{(z-l)\sqrt{(z-l)^2 + r^2} - (z+l)\sqrt{(z+l)^2 + r^2} + 4zl}{\left[(z+l) - \sqrt{(z+l)^2 + r^2}\right]\left[(z-l) - \sqrt{(z-l)^2 + r^2}\right]}$$

Table 1 – MAPLE code for the straight filamentary conductor with the finite length crossed by the electric current

The 2D and 3D magnetic field spectra (fig. 2, 3) and the 3D variation of the magnetic flux density in an axial section (fig. 4) were plotted on the basis of the obtained solutions. The values of the parameters are: electric current intensity $I = 100$ A, length $2l = 2$ m.

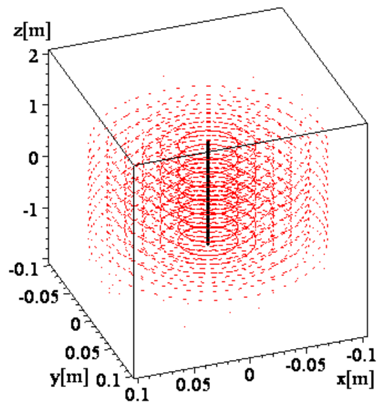

Figure 2: 3D spectrum for the magnetic field in the case of the straight filamentary conductor with the finite length.

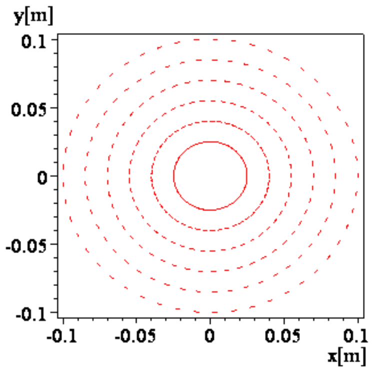

Figure 3: 2D spectrum for the magnetic field in the case of the straight filamentary conductor with the finite length in a cross section.

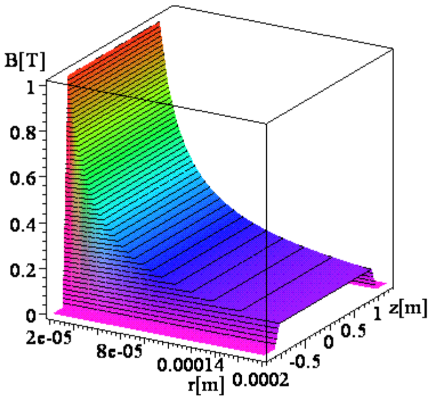

Figure 4: 3D variation of the magnetic flux density in an axial section in the case of the straight filamentary conductor with the finite length.

3.2. The magnetic field and the vector magnetic potential generated by the straight circular single turn crossed by the electric current (fig. 5)

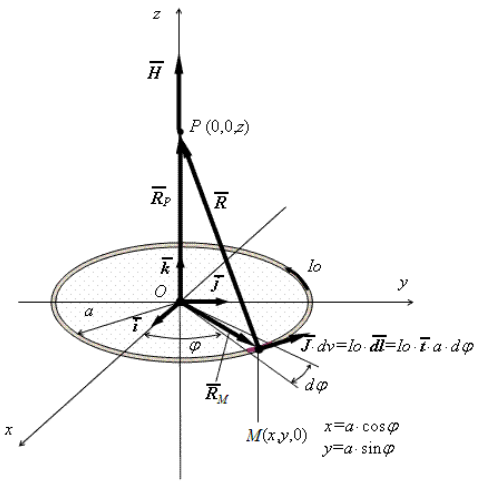

Figure 5: The straight circular single turn crossed by the electric current.

In the table 2 is given the MAPLE code used for the field computation in the case of the straight circular single turn.

```
> e[r]:=vector([1,0,0]);e[phi]:=vector(
[0,1,0]);k:=vector([0,0,1]);
> Rp:=r*e[r]+z*k;Rm:=z1*k;R:=Rp-Rm;
Rmod:=sqrt(R[1]^2+R[2]^2+R[3]^2);
> ds:=a*sin(phi)*e[r]+a*cos(phi)*e[phi];
>A=mu*Io/(4*Pi)*Int(ds/Rmod,phi=0..2*Pi)
>v:=[r,phi,z]:B:=curl([Ar,Aphi,Az],v,coo
rds=cylindrical);
```

$$B_r = \frac{\mu I_o}{2\pi r}\frac{z}{\left((r-a)^2+z^2\right)\sqrt{(r+a)^2+z^2}}\cdot\left[\left(r^2+a^2+z^2\right)\cdot\mathrm{EllipticE}\left(2\sqrt{\frac{ar}{(r+a)^2+z^2}}\right)-\left((r-a)^2+z^2\right)\cdot\mathrm{EllipticK}\left(2\sqrt{\frac{ar}{(r+a)^2+z^2}}\right)\right]$$

$$B_\varphi = 0$$

$$B_z = \frac{\mu I_o}{2\pi}\frac{1}{\left((r-a)^2+z^2\right)\sqrt{(r+a)^2+z^2}}\cdot\left[\left((r-a)^2+z^2\right)\cdot\mathrm{EllipticK}\left(2\sqrt{\frac{ar}{(r+a)^2+z^2}}\right)-\left(r^2-a^2+z^2\right)\cdot\mathrm{EllipticE}\left(2\sqrt{\frac{ar}{(r+a)^2+z^2}}\right)\right]$$

Table 2 – MAPLE code for the circular single turn.

The 3D magnetic field spectrum (fig. 6) and the 3D variations of the magnetic flux density in a parallel plane with the turn placed to a distance z and in an axial section (fig. 7, 8) were plotted on the basis of the obtained solutions. The values of the parameters are: electric current intensity $I = 100$ A, turn radius $a = 2$ cm.

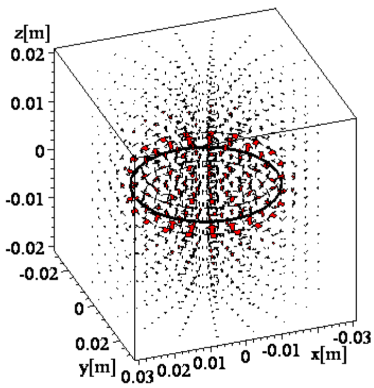

Figure 6: 3D spectrum for the magnetic field in the case of the straight circular single turn.

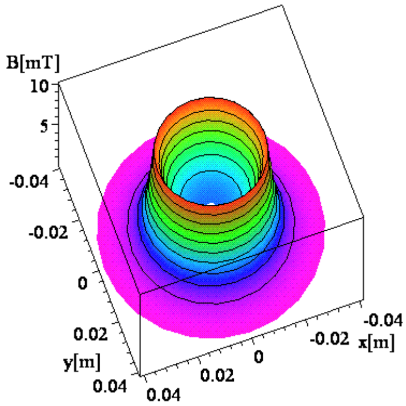

Figure 7: 3D variation of the magnetic flux density in a parallel plane with the turn placed to a distance z in the case of the straight circular single turn.

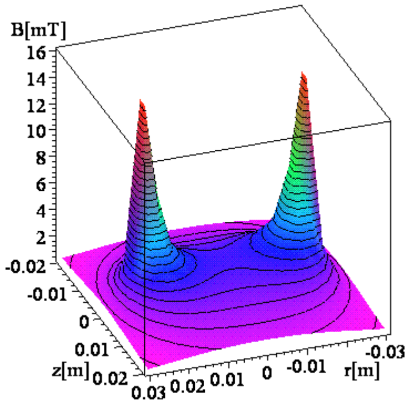

Figure 8: 3D variation of the magnetic flux density in an axial section in the case of the straight circular single turn.

3.3. The magnetic field and the vector magnetic potential generated by a straight bar with the rectangular cross-section and the infinite length crossed by the electric current (fig.9)

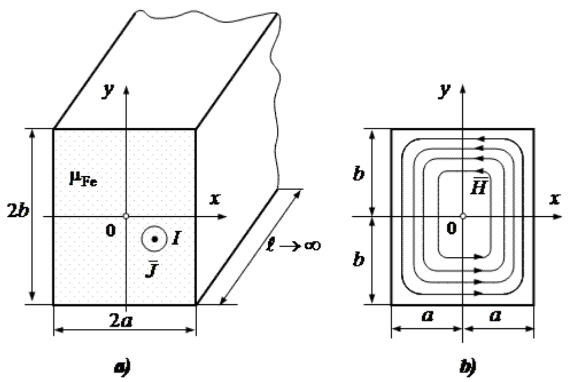

Figure 9: The straight bar with the rectangular cross-section and the infinite length crossed by the electric current.

In the table 3 is given the MAPLE code used for the magnetic field computation in the case of the straight bar with the rectangular cross-section and the infinite length.

```
>i:=vector([1,0,0]);j:=vector([0,1,0]);k
:=vector([0,0,1]);
> Rp:=x*i+y*j;Rm:=x1*i+y1*j;
Rp:=evalm(Rp); Rm:=evalm(Rm);R:=Rp-Rm;
R:=evalm(R);
> Rmod:=sqrt(R[1]^2+R[2]^2+R[3]^2);
J:=Io/(4*a*b)*k;
> dB:=mu/(2*Pi)*crossprod(J,R)/Rmod^2;
> dBx:=dotprod(dB,i);dBy:=dotprod(dB,j);
dBz:=dotprod(dB,k);
>Bx:=Int(Int(dBx,y1=-b..b),x1=-a..a);
By:=Int(Int(dBy,x1=-a..a),y1=-b..b);
Bz:=Int(Int(dBz,x1=-a..a),y1=-b..b);
```

$$B_x = -\frac{\mu I_O}{8\pi ab}\left[\frac{x+a}{2}\ln\frac{(x+a)^2+(y+b)^2}{(x+a)^2+(y-b)^2}+\right.$$
$$+\frac{x-a}{2}\ln\frac{(x-a)^2+(y-b)^2}{(x-a)^2+(y+b)^2}+$$
$$+(y-b)\left(\operatorname{arctg}\left(\frac{x-a}{y-b}\right)-\operatorname{arctg}\left(\frac{x+a}{y-b}\right)\right)+$$
$$\left.+(y+b)\left(\operatorname{arctg}\left(\frac{x+a}{y+b}\right)-\operatorname{arctg}\left(\frac{x-a}{y+b}\right)\right)\right]$$

$$B_y = \frac{\mu I_O}{8\pi ab}\left[\frac{y+b}{2}\ln\frac{(x+a)^2+(y-b)^2}{(x-a)^2+(y+b)^2}+\right.$$
$$+\frac{y-b}{2}\ln\frac{(x-a)^2+(y-b)^2}{(x+a)^2+(y-b)^2}+$$
$$+(x-a)\left(\operatorname{arctg}\left(\frac{y-b}{x-a}\right)-\operatorname{arctg}\left(\frac{y+b}{x-a}\right)\right)+$$
$$\left.(x+a)\left(\operatorname{arctg}\left(\frac{y+b}{x+a}\right)-\operatorname{arctg}\left(\frac{y-b}{x+a}\right)\right)\right]$$

$$B_z = 0$$

Table 3 –MAPLE code for the straight bar with the rectangular cross-section and the infinite length.

The magnetic field spectrum in a cross section (fig. 10) and the 3D variations of the magnetic flux density in a cross section (fig. 11) were plotted on the basis of the obtained solutions. The values of the parameters are: electric current intensity $I = 1000$ A, $2a = 8$ cm, $2b = 4$ cm.

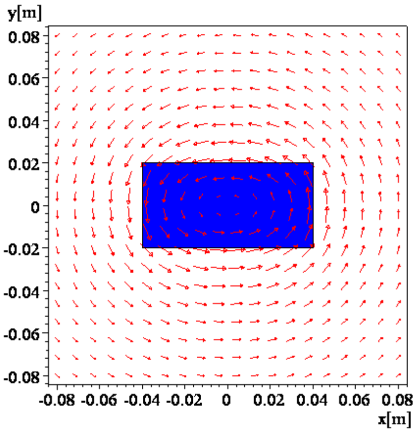

Figure 10: Magnetic field spectrum in the case of a straight bar with the rectangular cross-section and the infinite length.

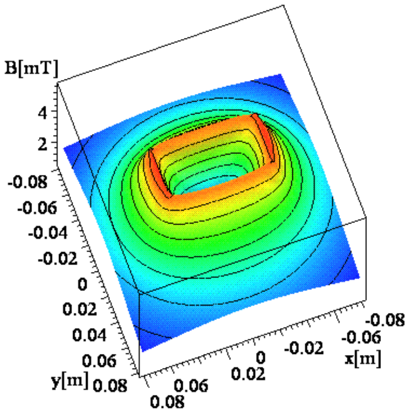

Figure 11: 3D variation of the magnetic flux density in the case of a straight bar with the rectangular cross-section and the infinite length.

4. CONCLUSIONS

The paper presents a new approach regarding the steady-state magnetic field computation, using symbolic analysis. This approach has the following advantages:

• the automatic writing of the general expressions (in any point from the space) of the magnetic field (or of the vector magnetic potential) by the adequate choice of the co-ordinates system (function of the problem symmetry) and the accurate calculation of these;

• the automatic drawing of the 2D and 3D magnetic field spectra, allowing that the suggestive images to be obtained;

• the development of the modeling skills, useful in the approach of others more complex problems;

• facilities in the treating of the limit cases (and of the degenerate cases, eventually);

• a better understanding of the physical phenomena corresponding to the analyzed field problem.

References

[1] C. I. Mocanu, *Teoria câmpului electromagnetic*, Ed. Didactică şi Pedagogică, Bucureşti, 1991

[2] Gh. Gavrilă, *Electrostatica*, Ed. Didactică şi Pedagogică, Bucureşti, 1998

[3] Gh. Gavrilă, *Bazele electrotehnicii vol. IV. Teoria câmpului electromagnetic (Electrocinetică şi Electrodinamică)*, Ed. Academiei Tehnice Militare, 1997

[4] I. Munteanu, D. Ioan, F. Enache, *Symbolic Computation in Electromagnetics: A Classroom Experiment*, Proceedings of CEFC'00, Milwaukee, Wisconsin, U.S.A., June 4-7, 2000

[5] Gh. Gavrilă, F. Enache, *Computation of the Electrostatic Fields in the Case of the True Electric Charges Distributions*, 5th International Conference on Electromechanical and Power Systems (SIELMEN 2005), October 6-8, 2005, Chişinău, Proceedings, volume 2, p.1004-1007

[6] D. Ioan, F. Enache, *Symbolic computation in electromagnetics*, Proceedings of 7th International Workshop Symbolic Methods and Applications to Circuit Design (SMACD 2002), Sinaia, Romania, October 10-11, 2002, pp.145-150

[7] K. M. Heal, M. L. Hansen, K. M. Richard, *Maple V – Learning Guide*, Ed. Springer Verlag, 1996