Abstract – The paper presents three procedures for the sensitivity analysis using auxiliary circuits: the Bykhovsky Perkins Cruz’s method, the incremental-circuit approach and the adjoint-circuit approach. All methods, based on the auxiliary circuits, have needed an efficient partially-symbolic method for the circuit analysis. For this, it was elaborated a very good procedure for the partially-symbolic analysis of the circuits based on the modified nodal equations and on the semi-state equations. This method was implemented in a program - SYCIAN – Symbolic Circuit Analysis. SYCIAN program can analysis together the original circuit and the auxiliary circuit for the complex linear time-invariant analog circuits. In this paper we will show that the methods of the sensitivity computation, based on the auxiliary circuits, become very competitive ones, if it is used the SYCIAN program. It is also underlined the advantages and disadvantages of the three procedures. Some illustrative examples are presented.

Keywords: Network functions, Lumped linear circuits, Sensitivity network, Symbolic analysis.

1. INTRODUCTION

In the design of any system (circuit), it is important to know the effect on the system performance due to the variations of some system (circuit) parameters. In the case of lumped, linear, time-invariant circuits, a precise measure of this effect can be expressed in terms of the sensitivity function to be defined next [1-4]. From a practical point of view, it is not sufficient for the design specification to be satisfied for a fixed set of nominal parameter values. A circuit designer should known the circuit performance is affected by changes in one or more parameter values. Any effect of the circuit function or any other circuit characteristic caused by a change in one or more circuit parameters is referred to as circuit sensitivity [1, 2, 4, 7, and 8]. Let \( F(s) \) be any circuit function of interest (driving-point immittances, transfer immittances, voltage gains, or current gains). At any particular frequency \( F \) is in general a complex number. The value of \( F \) of course varies with the frequency \( s = \omega \) in general, except for resistive circuits. Let \( x \) be any parameter associated with some circuit element, \( x \) may be the element value (such as the impedance or transadmittance) or some physical parameter (such as temperature or pressure) that affects the element value. The relative sensitivity, or simply the sensitivity of a circuit function \( F \), with respect to a parameter \( x \), denote by \( S_x^F \), is defined as

\[
S_x^F = \frac{\partial F}{\partial x}.
\]

According to equation (1), we may interpret the sensitivity \( S_x^F \) as the ratio of the fractional change in the circuit function \( F \) to the fractional change in the parameter \( x \), provided that all changes are sufficiently small (approaching zero theoretically). Sometimes we refer to \( S_x^F \) as the normalized sensitivity, in contrast with the unnormalized sensitivity, which is simply the partial derivative \( \partial F / \partial x \).

A circuit parameter \( x \) is called critical if the circuit sensitivity with respect to this parameter is very large. By using computer programs, the effects of parameter changes on the circuit performance can be predicted, allowing the circuit designer to select a circuit of low sensitivity in performance to one with higher sensitivity, while preserving the desired upper and lower limits of the performance function [1, 2, and 12].

The most important three reasons why parameter changes should be taken into account in circuit design are:

1. Parameter values of physical devices are not known exactly before-hand. There is always some discrepancy between the parameter values in a circuit model, which represents the physical circuit for computational purposes, and the exact parameter values. This is the problem of accuracy in circuit modeling and analysis.
2. During the lifetime of a manufactured circuit, parameters are subject to change through ageing and various environmental effects, such as ambient temperature and humidity. A sensitivity analysis is therefore required to find out which circuit parameter is critical.
3. The great spread of parameter values resulting from the circuit-manufacturing process requires the knowledge of the circuit performance in a certain range of parameter values, known as the tolerance range. This generates the need for tolerance analysis [1-4, 7-9].

There are several reasons for the great importance of network sensitivity in analog circuit design:
1. The study of circuit sensitivity enhances insight into circuit behavior, when changes in circuit parameters are involved. By dividing the circuit parameters into critical
and non-critical parameters, an effective method is provided to simplify circuit models for the purpose of more efficient circuit analysis.  
2. Knowledge of the circuit sensitivity can be used as a basis for comparing different electric circuits. It helps the circuit designer in selecting the proper circuit for a specified application.  
3. The effect of the manufacturing tolerance inherent in circuit elements can be investigated by sensitivity analysis. The notion of circuit sensitivity facilitates the development of methods for tolerance analysis.  
4. Network sensitivity plays an important part in the design and optimization of reliable circuits. In present, there are several computer-oriented methods for the calculation of sensitivities. Methods for the computation of multiparameter sensitivity are divided into three classes: 1. Methods based on feedback theory or the bilinear theorem for linear circuits; 2. Direct methods, by which the first-order derivatives of interest (and any desired higher-order derivatives) are computed directly from the circuit matrix, usually the nodal admittance matrix (or modified nodal matrix); 3. Indirect methods, which require an auxiliary circuit associated with the circuit under consideration. The most of the methods based on the auxiliary circuits have needed an efficient partially-symbolic method for the circuit analysis. We have elaborated a very good procedure for the partially-symbolic analysis of the circuits based on the modified nodal equations and on the semi-state equations.[4-6, 9-12]. This method was implemented in a program - SYCICAN – Symbolic Circuit Analysis [9]. SYCICAN program can analysis together the original circuit and the auxiliary circuit. In this paper we will show that the methods of the sensitivity computation, based on the auxiliary circuits, become very competitive ones, if it is used the SYCICAN program.  

2. SENSITIVITY ANALYSIS USING AUXILIARY CIRCUITS  

2.1. Bykhovski Perkins Cruz’s method  

One significant method, first enunciated by Bykhovski [1,7], was described by Perkins and Cruz [7] – called Bykhovsky-Perkins-Cruz’s method. Let be a linear and time-invariant circuit C. An auxiliary circuit can be constructed by replacing independent voltage sources by short-circuits, removing independent current sources and applying a controlled source in the branch of the element x with respect to which the network sensitivity is desired. The auxiliary circuit C_x has the same graph as the original circuit C. In the auxiliary circuit the branch currents and branch voltages represent the derivatives \( \frac{\partial i_j}{\partial x} \) and \( \frac{\partial v_j}{\partial x} \) respectively. The auxiliary circuit thus obtained will be called the sensitivity circuit. Let us consider an RLC network containing \( n_R \) resistors, \( n_C \) capacitors and \( n_L \) inductors. The following branch relations hold:

\[
\begin{align*}
    u_k &= R_k i_k, \quad k = \frac{1}{R_k}; i_{ic} = C_i \frac{d v_c}{dt}, \\
    j &= \frac{1}{\omega_C}; u_j = L_p \frac{d i_p}{dt}, \quad p = \frac{1}{\omega_L}.
\end{align*}
\]

If we assume that the parameter x be the resistance \( R_k \) of a resistor, we can write

\[
\begin{align*}
    \frac{\partial u_k}{\partial R_k} &= i_k + R_k \frac{\partial i_k}{\partial R_k}, \quad C_i \frac{d \left( \frac{\partial v_c}{\partial R_k} \right)}{dt}, \\
    u_{t_p} &= L_p \frac{d \left( \frac{\partial i_p}{\partial R_k} \right)}{dt}.
\end{align*}
\]

According to the relations (3) all the above partial derivatives of voltages and currents with respect to \( R_k \) are regarded as voltages and currents of a new circuit, the sensitivity circuit, its branch relations will be similar to those of the original controlled voltage, except in the Branch \( R_k \) which is augmented with a controlled voltage source in series (as in Table 1).

<table>
<thead>
<tr>
<th>The circuit element x.</th>
<th>Branch structure in C_x.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear resistor linear, ( R_k )</td>
<td>( i_k = \frac{\partial u_k}{\partial R_k} )</td>
</tr>
<tr>
<td>Linear capacitor, ( C_i )</td>
<td>( \frac{\partial v_c}{\partial C_i} )</td>
</tr>
<tr>
<td>Linear inductor, ( L_k )</td>
<td>( \frac{\partial i_p}{\partial L_k} )</td>
</tr>
<tr>
<td>CCVS ( e_v(u_C) ), ( R_{e-C} )</td>
<td>( \frac{\partial v}{\partial u_C} )</td>
</tr>
<tr>
<td>VCCS ( j_v(u_C) ), ( G_{e-C} )</td>
<td>( \frac{\partial i}{\partial u_C} )</td>
</tr>
<tr>
<td>VCVS ( e_C(u_C) ), ( A_{e-C} )</td>
<td>( \frac{\partial v}{\partial u_C} )</td>
</tr>
<tr>
<td>CCCS ( j_v(u_C) ), ( B_{e-C} )</td>
<td>( \frac{\partial i}{\partial u_C} )</td>
</tr>
</tbody>
</table>

Table 1: Construction of the auxiliary circuit.
The two circuits (the original circuit and auxiliary circuit) will be analyzed together having the same datum node (see the example).

Figure 1. a) Original circuit \( C \); b) Auxiliary circuit \( C_a \).

Let the circuit of Fig. 1, a be given. It is required to find the first-order sensitivity \( S_d(\mu_{13}, C_{12}) \). The sensitivity circuit is shown in Fig. 1, b.

Running the SYCian program [4, 9] we obtain the following results:

\[ S_{A4,6,1,17,C12} = U_{28} = 0.1200 \times 10^{11} \frac{s}{(4000.0 + s)^2} , \]

The desired sensitivity \( \frac{\partial U_{13}}{\partial C_{12}} = S_d(\mu_{13}, C_{12}) \) is equal to the voltage \( U_{28} \) from the circuit in Fig. 1, b

\[ \frac{\partial U_{13}}{\partial C_{12}} = U_{28} = \frac{\partial A_{4,6,1,17}}{\partial C_{2}} = \frac{1.2 \times 10^{10}}{(s + 4000.0)^2} \] (4)

2.2. Incremental circuit approach

Let us consider a linear circuit \( C \) consisting of a number of standard branches. We shall vary the impedance of each branch by a slight amount and obtain a perturbed circuit \( C_p \). For \( C \), we can write

\[ \text{KCL: } A I_b = 0 ; \quad \text{KVL: } B U_b = 0 , \] (5)

where \( A \) and \( B \) is the reduced incidence matrix and fundamental loop matrix, respectively, and \( I_b \) and \( U_b \) are the branch current vector and branch voltage vector, respectively. Since the perturbed circuit \( C_p \) has the same topology as \( C \), for \( C_p \) we have

\[ \text{KCL: } A(I_b + \Delta I_b) = 0 ; \quad \text{KVL: } B(U_b + \Delta U_b) = 0 . \] (6)

From equations (5) – (6), we immediately have

\[ A \Delta I_b = 0 ; \quad B \Delta U_b = 0 , \] (7)

which indicate that the incremental currents \( \Delta I_b \) and incremental voltages \( \Delta U_b \) have the same constraints as \( I_b \) and \( U_b \). Therefore, \( \Delta I_b \) and \( \Delta U_b \) could possibly be the branch currents and voltages of some circuit \( C_i \) having the same topology as \( C \), provided that the branch characteristics of \( C_i \) are properly defined. We shall investigate how branch characteristics of \( C_i \) are to be defined for the purpose of producing \( \Delta I_b \) and \( \Delta U_b \). Consider an element in \( C \) with impedance \( Z \), then

\[ U = Z I. \] (8)

For the perturbed circuit, the same element is described by

\[ I_Z = Z I + \Delta Z \Delta I \] (9)

From equations (8) and (9), we have

\[ \Delta U = Z \Delta I + \Delta I \Delta Z \] (10)

Provided that \( \Delta Z \) (and hence \( \Delta I \) and \( \Delta U \)) is infinitesimally small, we can neglect higher order terms and rewrite equation (10) as

\[ \Delta U = Z \Delta I + I \Delta Z \] (11)

which indicates that in \( C \), the branch having \( \Delta U, \Delta I \) consist of a impedance \( Z \) (the original impedance in \( C \)) in series with a current-controlled voltage sources \( I \Delta Z \). This is illustrated in Table 2.
The incremental circuit \( C_1 \), a has nominal values represented in this figure. The active circuit must be analyzed together (having the same datum-quantities from the original circuits, the two circuits the incremental circuit are controlling by the neglected. Because all the controlled sources from the original circuit are controlling by the second-order effects can be neglected. Because all the controlled sources from the incremental circuit are controlling by the quantities from the original circuits, the two circuits must be analyzed together (having the same datum-node) [4, 9 – 12]. The active circuit \( C \) shown in Fig. 1, a has nominal values represented in this figure. The incremental circuit \( C_i \) is shown in Fig. 2. Find the partial derivatives of the voltage gain \( A_{2,3,1,11} \) with respect to the circuit parameters.

<table>
<thead>
<tr>
<th>Table 2: Construction of the incremental circuit.</th>
</tr>
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The circuit \( C_i \) derived from \( C \) according to the rules of Table 1.2, is an incremental circuit, since branch currents and voltages in \( C_i \) are \( \Delta I \)'s and \( \Delta U \)'s. It should be emphasized that the rules in Table 2 for the construction of the incremental circuit are derived under the assumption that all parameter changes are very small and that second-order effects can be neglected. Because all the controlled sources from the incremental circuit are controlling by the quantities from the original circuits, the two circuits must be analyzed together (having the same datum-node) [4, 9 – 12]. The active circuit \( C \) shown in Fig. 1, a has nominal values represented in this figure. The incremental circuit \( C_i \) is shown in Fig. 2. Find the partial derivatives of the voltage gain \( A_{2,3,1,11} \) with respect to the circuit parameters.

![Figure 2. Sensitivity calculation by the incremental-circuit method.](image)

Because, the input voltage \( E_i = 1 \text{ V} \) then \( A_{4,6,1,17} = U_{13} \) and \( dA_{4,6,1,17} = du_{28} \left( u_{28} \right) \).

Running the SYCIAN program [9] we obtain:

\[
d_{A4,6,1,17} = 9 \times (-4.5 \times dR14 + 7.5 \times dR6 + 3.0 \times dR11 + 2500.0 \times dA7_825000000.0 \times dG9_10 + 12e11 \times dC12 + 1.2e-3 \times dB2_3 \times s^2 + 4.5e-2 \times dR4_5 + 2.5 \times dA7_8 \times s + 0.45e-2 \times dR14 \times s)
\]

and

\[
\frac{\partial u_{13}}{\partial C_{12}} = \frac{\partial A_{4,6,1,17}}{\partial C_{2}} = \frac{1.2 \times 10^{10}}{(s + 4000.0)^2}
\]

The sensitivity \( \frac{\partial A_{4,6,1,17}}{\partial C_{12}} \) is identical to the one obtained by the Bykhovsky-Perkins-Cruz’s method.

### 2.3. Adjoint circuit approach

Two linear time-invariant circuits \( C \) and \( \hat{C} \) are adjoint circuits of each other if the following three conditions are satisfied:

1. Both circuits have the same topology; i.e., \( A = \hat{A} \) and \( B = \hat{B} \).

For controlled sources, we consider a controlling voltage as that across an open-branch (an ideal independent current source with \( j = 0.0 \text{ A} \)), and a controlling current as that through a short-circuit branch (an ideal independent voltage source with \( e = 0.0 \text{ V} \)).

2. If the no independent-source branches of \( C \) and \( \hat{C} \) possess branch impedance matrices \( Z_b \) and \( \hat{Z}_b \), respectively, then

\[
Z_b^1 = \hat{Z}_b^1
\]

On the other hand, if branch-admittance matrices \( Y_b \) and \( \hat{Y}_b \) exist, then

\[
Y_b^1 = \hat{Y}_b^1
\]

In general case, the no source branches of \( C \) and \( \hat{C} \) can always be characterized by hybrid matrices \( \ddot{H}_b \) and \( \ddot{H}_b \) thus:

\[
\begin{bmatrix}
I_{b1} \\
U_{b2} \\
\end{bmatrix} = \begin{bmatrix}
Y_{b11} & B_{b12} \\
A_{b21} & Z_{b22} \\
\end{bmatrix} \begin{bmatrix}
U_{b1} \\
I_{b2} \\
\end{bmatrix} = \ddot{H}_b \begin{bmatrix}
U_{b1} \\
I_{b2} \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\dot{I}_{b1} \\
\dot{U}_{b2} \\
\end{bmatrix} = \begin{bmatrix}
\dot{Y}_{b11} & \dot{B}_{b12} \\
\dot{A}_{b21} & \dot{Z}_{b22} \\
\end{bmatrix} \begin{bmatrix}
\dot{U}_{b1} \\
\dot{I}_{b2} \\
\end{bmatrix} = \ddot{H}_b \begin{bmatrix}
\dot{U}_{b1} \\
\dot{I}_{b2} \\
\end{bmatrix}
\]

For \( C \) and \( \hat{C} \) to be adjoint circuits of each other, we require that

\[
\begin{bmatrix}
\dot{Y}_{b11} & \dot{B}_{b12} \\
\dot{A}_{b21} & \dot{Z}_{b22} \\
\end{bmatrix} = \begin{bmatrix}
Y_{b11} & -A_{b12} \\
-B_{b12} & Z_{b22} \\
\end{bmatrix}
\]

3. Corresponding independent sources in both circuits are the same in nature (current or voltage sources), but need not have the same values.
In many applications of adjoint circuits, it is sometimes convenient to extract all independent sources a multiport. We denote the port currents and voltages by $I_p$ and $V_{Jo}$ respectively. In [1, 2] it is proved that if the open-circuit admittance matrices $Y_{oc}$ and $Z_{oc}$ (the short-circuit impedance matrices) exist for the multiport created from $C$ and its adjoint $\hat{C}$, then

$$Z_{oc}^1 = \hat{Z}_{oc}, \quad Y_{sc}^1 = \hat{Y}_{sc}. \quad (17)$$

It is more convenient to construct the adjoint circuit $\hat{C}$ from any given circuit $C$ by use of Table 3.

Table 3: Construction of the adjoint circuit.

<table>
<thead>
<tr>
<th>Original circuit, $C$</th>
<th>Adjoint circuit, $\hat{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Original Circuit 1]</td>
<td>![Adjoint Circuit 1]</td>
</tr>
<tr>
<td>![Original Circuit 2]</td>
<td>![Adjoint Circuit 2]</td>
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<tr>
<td>![Original Circuit 3]</td>
<td>![Adjoint Circuit 3]</td>
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<tr>
<td>![Original Circuit 4]</td>
<td>![Adjoint Circuit 4]</td>
</tr>
<tr>
<td>![Original Circuit 5]</td>
<td>![Adjoint Circuit 5]</td>
</tr>
<tr>
<td>![Original Circuit 6]</td>
<td>![Adjoint Circuit 6]</td>
</tr>
</tbody>
</table>

The multiport create from $C$ and $\hat{C}$ by extracting all independent sources can be characterized by hybrid matrices $H$ and $\hat{H}$, respectively, as follows:

$$\begin{bmatrix} I_E \\ U_J \end{bmatrix} = \begin{bmatrix} Y_{EE} & B_{JE} \\ A_{JE} & Z_{EE} \end{bmatrix} \begin{bmatrix} U_E \\ I_J \end{bmatrix} = H \begin{bmatrix} U_E \\ I_J \end{bmatrix} \quad (18)$$

and

$$\begin{bmatrix} I_E \\ U_J \end{bmatrix} = \begin{bmatrix} \hat{Y}_{EE} & \hat{B}_{JE} \\ \hat{A}_{JE} & \hat{Z}_{EE} \end{bmatrix} \begin{bmatrix} \hat{U}_E \\ \hat{I}_J \end{bmatrix} = \hat{H} \begin{bmatrix} \hat{U}_E \\ \hat{I}_J \end{bmatrix}, \quad (19)$$

where the subscript $E$ indicates independent voltage sources and $J$ indicates independent current sources. It can be shown that matrices $H$ and $\hat{H}$ are related in the following manner:

$$\begin{bmatrix} \hat{Y}_{EE} & \hat{B}_{JE} \\ \hat{A}_{JE} & \hat{Z}_{EE} \end{bmatrix} = \begin{bmatrix} Y_{EE} & A_{JE} \\ Z_{EE} \end{bmatrix}^{1}.$$

(20)

An equation relating the changes in $H$ to the changes in $H_b$ can be derived [1, 2]. This relationship has the following form:

$$-\hat{I}_b^1 \Delta \hat{U}_j + \hat{U}_b^1 \Delta E = \begin{bmatrix} \Delta Y_{EE} & \Delta B_{JE} & U_E \\ \Delta A_{JE} & \Delta Z_{EE} & I_J \end{bmatrix} = \begin{bmatrix} -\hat{U}_b^1 \\ \hat{I}_b^1 \end{bmatrix}.$$

(21)

For different types of elements in $C$ and $\hat{C}$, the right side of equation (21) may be evaluated separately [1 – 9]. The procedure for calculating $\partial V_o / \partial x_1$, $\partial V_o / \partial x_2$, or $\partial h_{j1} / \partial x_k$ may by summarized as follows>

Step 1. Perform an analysis of the circuit $C$ to obtain $V_{b1}$ and $I_{b2}$.

Step 2. Select the excitations for $\hat{C}$ such that one side of equation (20) yields only one term, which is $\Delta V_o$, $\Delta I_o$, or $\Delta h_{j1}$, of interest. Perform an analysis of the adjoint circuit $\hat{C}$ to obtain $\hat{V}_{b1}$ and $\hat{I}_{b2}$.

Step 3. Evaluate the right side of equation (21), either directly from matrix multiplications or with the aid to Table 4. From the resultant expression, obtain the desired partial derivatives.

Consider the circuit shown in Fig. 3, a. Find the partial derivatives of $\Delta A_{JE}$ with respect to all parameters of the circuit in Fig. 3. An examination of equation (20) shows that to have $\Delta A_{JE}$ only, we may choose $U_E = E_{15} = 1 \text{ V}$, $I_J = J_{10} = 0 \text{ A}$ and $\hat{U}_E = \hat{E}_{15} = 0 \text{ V}$, $\hat{I}_J = \hat{J}_{13} = -1 \text{ A}$.

For example, the partial derivative of the voltage gain $A_{3,4,1,5}$ with respect to the parameter $B_{2,3}$ has the following expression:
The correctness of these results may be verified for this circuit by evaluating the following expression at the nominal element values:

\[ S_{A_{2,3}}^{A_{4,6,1,9}} = \frac{\partial A_{4,6,1,17}}{\partial B_{2,3}} = \frac{0.00012 \cdot S^3}{(s + 4000.0)^2}. \]

The adjoint-network approach can be used to derive second-order derivatives of linear, time-invariant circuits [1]. The adjoint-network concept has been used to perform circuit analysis when large parameter changes occur in the circuit [2, 8]. It can also be used for locating faults and for selecting test point with the purpose of locating faults in linear and nonlinear analog circuits [1 - 4].

3. CONCLUSION

The three methods for the sensitivity calculations, presented in this paper, can be applied to a large class of linear circuit containing: all four types of linear controlled sources, resistors, inductors, capacitors, nullors, and any multiterminal or multiport circuit element having an equivalent scheme made up only by two-terminal elements and controlled sources.

All sensitivity methods based on the auxiliary circuits have need of a very good procedure for partial-symbolic analysis. Our symbolic analysis technique, based on the modified nodal equations and implemented into a computing program, is suitable to analysis the original circuit together the auxiliary circuit. In this way, the sensitivity computation becomes more efficient. Higher-order derivatives can be easily computed by attaching new higher-order sensitivity circuits to lower-order sensitivity networks. One of the disadvantages of the B - P - C’s method is that in implementing of this method, the sensitivity network required for each circuit parameter of interest is coupled with the original circuit by inserting an appropriate controlled source. A large number of parameters \( x \) would make the size of the circuit that could be handled prohibitive. However, this method allows all frequency and time responses of both currents and voltages and their sensitivities to parameter changes to be evaluated in one computer run. From point of view of the computational effort, the incremental-circuit approach is less efficient than the adjoint-circuit approach. However, the incremental-circuit method has the following advantages: it provides better insight the effect of parameter variations, and in the analysis of the incremental network, the incremental currents and voltages of all branches are available from the calculations. Such incremental quantities are themselves of interest in some applications.

References