Estimation of the Heat Cumulated Inside a Wall with Cylindrical Symmetry

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Abstract – **The issue of mathematical modeling in order to simulate physical processes or physical systems, permanently preoccupy a significant number of researchers. The sustained efforts being made in this direction are determined by the benefits that simulation can provide and by the development of software packages capable of performing such simulations. It is obvious that mathematical modeling followed by simulation allows the study of possible situations that would be difficult to achieve in real systems and could lead to an irremediable defect. If the mathematical modeling of stationary processes with respect to time is pretty well set, for the transient process simulation is still much to study. The mathematical equations describing such processes are generally of higher order, sometimes they are transcendent, and thus harder to solve. In the paper the authors are concerned with the modeling of transient thermal processes. When analyzing electrical heating systems, the main problem occurs due to heat transfer through the walls. This phenomenon appears at multilayered cylindrical walls, when heat is transferred through it, and a part of it is stored into the wall as internal energy. This paper aims to determine some mathematical relationships which describe the heat storage processes in components with cylindrical geometry. Determined mathematical equations will be used furthermore in the modeling of transient thermal processes in structures with cylindrical geometry. This allows the simulation of multilayer insulation systems under a transient thermal regime for a long time, using Matlab-Simulink toolkit.**

Keywords: *cylindrical symmetry, energy balance, heat transfer, stored energy.*

INTRODUCTION

One of characteristics of the modern times is represented by a large computer use in most various fields. This is due, mainly, to increased speed and computing power by the improvement of hardware structures. But equally important is the quick development of some software packages that enable the development of numerical simulation software dedicated to different applications

A simulation performed correctly has the advantage that can be found optimal solutions even before it starts the physical design of a plant. Also, through simulation it can be determined the failure modes which, in many cases, can cause destructive effects on plants. So, during the design, one can take the necessary steps to avoid their occurrence.

In the simulation of technical processes researchers are often faced with problems that are difficult to be mathematically modeled. Some other times mathematical models become extremely difficult. If for steady-state phenomena the mathematical models and simulations are quite well developed, in transient regime things get complicated. Often these regimes are carried out quickly, especially if one wants to achieve a plant driven in real-time by a computer system. This is one of the reasons by which mathematical models that are based to the simulation should have a simple structure.

The use of simplified equations, but which obey physical phenomena, has allowed modeling and simulation with a good accuracy of the transient heat transfer through a plane multilayer wall (e.g. the wall of a furnace for heat treatment). These results were partially presented in [1] and [2]. Figure 1 shows the Matlab Simulink model of a plane-parallel multilayer wall of a furnace, and in Figure 2 is detailed the Simulink model of the intermediate layer. Note that the model has been designed to allow very easy insertion of one or more interlayers, based on the same modeling structure. In this case it was considered that, in the modeling and simulation, the temperature in each layer composing the wall modifies by a linear law. The temperature from the middle of that layer represents the average of temperatures from its sides (interior and exterior) [6, 9, 10]. The linear distribution of the temperatures inside the layer of the wall allowed rapid computing of the heat accumulated or disposed by the wall between two consecutive time moments, belonging to the transitory process of heating respectively of cooling the furnace.

On the other hand, there are many cases to be mathematically modeled and possibly simulated the heat transfer through the layers of multilayer walls with cylindrical symmetry. Experience has led to the desire to determine mathematical relationships as simple, that could characterize the physical phenomena of the cylinder walls and that could be used similar to those that led to the modeling of plane walls.

This paper aims to find a simple way to mathematical modeling of thermal systems with cylindrical structure, based on which to build future simulation software of multilayered cylindrical thermal insulation commonly found in electro-thermal plants or in energy transport cables.

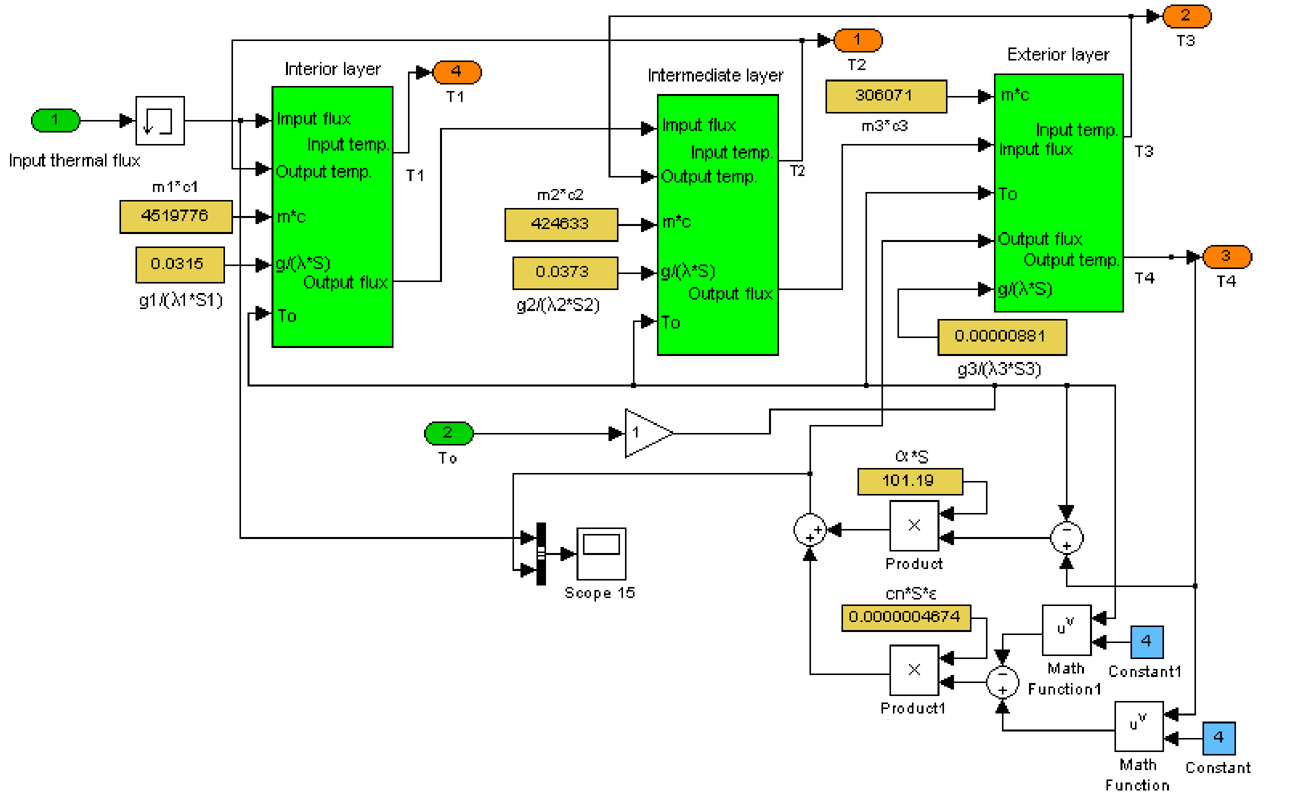

Fig.1. The block diagram of the wall.

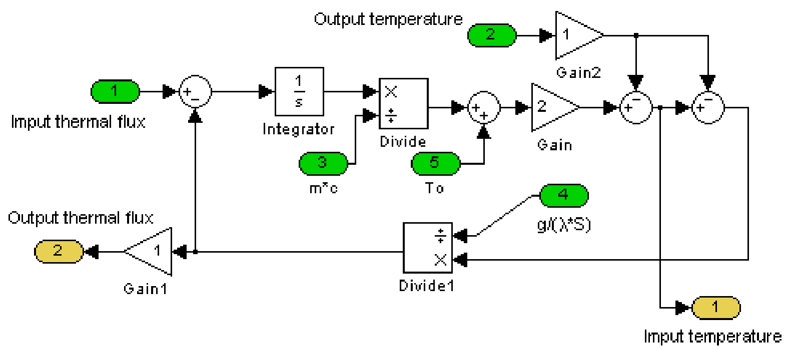

Fig.2. The thermo-insulated layer.

PRELIMINARY

In the transient modeling and simulation, it should be pointed out that each layer of the composite wall will accumulate or provide different thermal energy from the heat flow that crosses it. The stored energy is proportional with the mass, specific heat of the layer and the temperature difference measured between two successive time moments. For the entire volume of the wall, the stored energy can be calculated:

$$Q_s = \int_V \left(m_v \cdot c_{pv} \cdot \Delta\theta_v\right) \mathrm{d}v \quad [\mathrm{J}] \qquad (1)$$

where:

m_v = the unit volume mass [kg];

c_{pv} = specific heat in the unit volume [J/(kg·K)];

$\Delta\theta_v$ = the temperature difference in the unit volume [K].

The problem during modeling and simulation of this simple formula is to determine the change of the temperature field inside the wall. This determination must be conducted with enough high speed, in order to allow its use in a real-time system for monitoring and control.

For the plane-parallel wall, crossed by a transverse heat flux, at which internal temperatures of each layer are distributed linearly between the temperatures of the two sides, the average temperature is computed as the arithmetic average of the temperatures of the sides [1, 2].

In order to achieve a faster computation, for the homogenous layers ($c_p = const.$), it is preferable to determine an equivalent increase of temperature ($\Delta\theta_{ech}$), which then will be used to determine the stored heat.

$$Q_s = m_{tot} \cdot c_p \cdot \Delta\theta_{ech} \quad [\text{J}] \qquad (2)$$

where:

m_{tot} = the mass of entire wall [kg].

The problem becomes more delicate when discussing about the heat transfer through a composite tubular wall, thus based on a cylindrical model.

In this case the thermal diffusion equation will be written in cylindrical coordinates [4, 7]:

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(\lambda\cdot r\cdot\frac{\partial\theta}{\partial r}\right)+\frac{1}{r^2}\cdot\frac{\partial}{\partial\varphi}\left(\lambda\cdot\frac{\partial\theta}{\partial\varphi}\right)+ \\ +\frac{\partial}{\partial z}\left(\lambda\cdot\frac{\partial\theta}{\partial z}\right)+w_q=\rho\cdot c_p\cdot\frac{\partial\theta}{\partial t} \qquad (3)$$

where:

- λ = thermal conductivity [W/(m·K)];
- w_q = the rate at which energy is generated per unit of volume of the medium [W/m^3];
- $\rho\cdot c_p\cdot\frac{\partial\theta}{\partial t}$ = the time rate of change of the thermal energy of the medium per unit of volume [W/m^3];

The paper presents a method of determining the representative layer (radius) within a tubular insulator. Its temperature can be used to compute the thermal energy stored in the entire insulating layer.

It is envisaged a single insulating, cylindrical layer, with length L , crossed by a radial heat flow (Fig. 3).

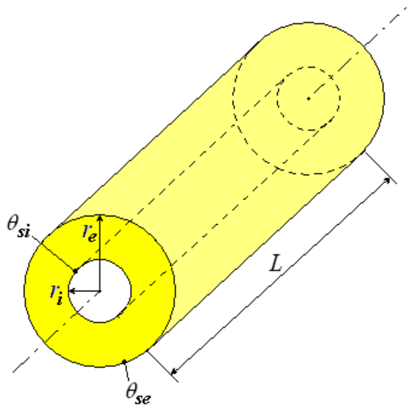

Fig.3. Cylindrical layer.

According to [3], in steady-state conditions, considering that within the insulating material there are no sources of heat, equation (3) becomes:

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(\lambda\cdot r\cdot\frac{\partial\theta}{\partial r}\right)=0 \qquad (4)$$

Considering $\theta_{si} > \theta_{se}$, the heat current (constant in steady-state regime) that will cross the insulating layer, oriented from the inside to outside, will be equal to:

$$\dot{Q}_r = \frac{\mathrm{d}Q_r}{\mathrm{d}t} = \frac{\Delta\theta}{R_{t-cil}} = \frac{2\pi\cdot L\cdot\lambda\cdot(\theta_{si}-\theta_{se})}{\ln(r_e/r_i)} \quad [\text{W}] \qquad (5)$$

where:

L = the cylindrical layer length [m];
r_e = outside radius [m];
r_i = inside radius [m].
R_{t-cil} = thermal resistance of the hollow cylinder [K/W]

For λ = const., equation (4) will have the solution:

$$\theta_r = C_1\cdot\ln(r)+C_2$$

The constants are determined considering the boundary temperature conditions:

$$\begin{cases}\theta(r_i)=\theta_{si}\\ \theta(r_e)=\theta_{se}\end{cases}$$

Finally we get the following formula for the temperature, also described in [5]:

$$\theta_r = \frac{\theta_{si}-\theta_{se}}{\ln(r_i/r_e)}\cdot\ln\left(\frac{r}{r_e}\right)+\theta_{se} \quad [\text{K}] \qquad (6)$$

Relation (6) indicates that in each moment the temperature in any point inside the wall (at distance r from the axis) depends on the temperature difference between the two outer surfaces. During the transient process at least one parameter shall be modified.

DETERMINING THE EQUIVALENT COMPUTATIONAL RELATIONSHIPS

When determining the amount of energy stored in the cylindrical insulation wall, it will be considered an elementary cylindrical volume, very thin, with the inner radius $(r-\mathrm{d}r/2)$, the outer radius $(r+\mathrm{d}r/2)$ and length L . The elementary volume considered for $L = 1\text{m}$, will be:

$$V_{elm} = L\cdot\left(\pi(r+\mathrm{d}r/2)^2-\pi(r-\mathrm{d}r/2)^2\right)=2\pi\cdot r\cdot\mathrm{d}r \qquad (7)$$

where:

r = computational radius $r\in[r_i, r_e]$ [m]

Since the elementary cylinder is very thin ($\mathrm{d}r\to 0$), it can be considered as limit that, in a certain moment, the temperature is defined throughout its entire volume by equation (6). If we consider the initial temperature of the whole insulating coating as being equal to the ambient temperature (θ_0), it can define the warming of the outer surfaces also in the elementary volume:

$$\begin{cases}\Delta\theta_{si}=\theta_{si}-\theta_0\\ \Delta\theta_{se}=\theta_{se}-\theta_0\\ \Delta\theta_r=\theta_r-\theta_0\end{cases} \qquad (8)$$

If we modify equation (6) as:

$$\theta_r-\theta_0 = \frac{\theta_{si}-\theta_{se}+\theta_0-\theta_0}{\ln(r_i/r_e)}\cdot\ln\left(\frac{r}{r_e}\right)+\theta_{se}-\theta_0$$

we obtain the formula for calculating the heating of the particles in the elementary volume:

$$\Delta\theta_r = \frac{\Delta\theta_{si} - \Delta\theta_{se}}{\ln(r_i/r_e)} \cdot \ln\left(\frac{r}{r_e}\right) + \Delta\theta_{se} \tag{9}$$

The energy stored in the elementary volume:

$$Q_{elm} = m_{elm} \cdot c_p \cdot \Delta\theta_r = \rho \cdot c_p \cdot V_{elm} \cdot \Delta\theta_r = \tag{10}$$

$$= 2(\pi \cdot \rho \cdot c_p) \cdot r \cdot \mathrm{d}r \cdot \left[\frac{\Delta\theta_{si} - \Delta\theta_{se}}{\ln(r_i/r_e)} \cdot \ln\left(\frac{r}{r_e}\right) + \Delta\theta_{se}\right]$$

This is used to compute the part of the heat flow which is stored as heat in the whole insulating cylinder:

$$Q_s = \int_{r_i}^{r_e} 2(\pi \cdot \rho \cdot c_p) \cdot r \cdot \mathrm{d}r \cdot \left[\frac{\Delta\theta_{si} - \Delta\theta_{se}}{\ln(r_i/r_e)} \cdot \ln\left(\frac{r}{r_e}\right) + \Delta\theta_{se}\right] \tag{11}$$

Calculations finally lead to the value:

$$Q_s = 2(\pi \cdot \rho \cdot c_p) \cdot \left[\frac{\Delta\theta_{si} - \Delta\theta_{se}}{\ln(r_i/r_e)} \left(\frac{r_i^2}{2} \ln\left(\frac{r_e}{r_i}\right) - \frac{1}{4}\left(r_e^2 - r_i^2\right)\right) + \frac{\Delta\theta_{se}}{2}\left(r_e^2 - r_i^2\right)\right] \tag{12}$$

Equation (11) implemented in a numerical simulation system, involves making a large number of calculations for each simulation step [8].

In order to increase the simulation speed we will try to determine an equivalent heating (for the entire cylindrical insulating layer of unitary length). Considering the value of the equivalent heating as being equal to the heating of the particles at distance x from the axis ($\Delta\theta_{ech} = \Delta\theta_x$), according to equation (2) is obtained:

$$Q_s = \pi \cdot \left(r_e^2 - r_i^2\right) \cdot \rho \cdot c_p \cdot \Delta\theta_x \tag{13}$$

Equivalence in terms of energy implies that formulas (12) and (13) lead to the same result and allows us (after equivalence and calculations) to determine the equivalent heating:

$$\Delta\theta_x = \Delta\theta_{se} - \frac{\Delta\theta_{si} - \Delta\theta_{se}}{2\ln(r_i/r_e)} + \frac{r_i^2}{\left(r_e^2 - r_i^2\right)} \cdot \left(\Delta\theta_{se} - \Delta\theta_{si}\right) \tag{14}$$

Using the transformations defined by (8), we will get the temperature at distance x from the axis:

$$\theta_x = \theta_{se} + K \cdot \left(\theta_{si} - \theta_{se}\right) \tag{15}$$

where constant K can be determined based on the geometry of the cylindrical insulating layer:

$$K = \frac{1}{2 \cdot \ln(r_e/r_i)} - \frac{1}{(r_e/r_i)^2 - 1} \tag{16}$$

If in (6) it is considered $r = x$ and θ_r is replaced by θ_x from (15), we can determine the distance measured from the axis of the cylinder, where the particles were heated with $\Delta\theta_{ech}$. This value is:

$$x = \frac{r_e}{\sqrt{e}} \cdot \left(\frac{r_e}{r_i}\right)^{\frac{r_i^2}{r_e^2 - r_i^2}} \tag{17}$$

The x may be determined based on the geometric dimensions of the intermediate layer.

THE ANALYSIS OF RESULTS

Graphical representation of the results expressed by equations (15) and (17) allow verifying their correctness and highlighting some important conclusions. For this, the sizes x/r_e and K are plotted as functions of the ratio r_e/r_i.

The following figures show such representations for different areas of variation of the amount of the rays r_e/r_i.

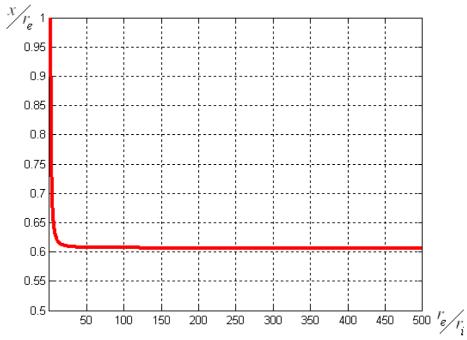

Fig.4. Equivalent radius x for $[0<r_e/r_i<500]$.

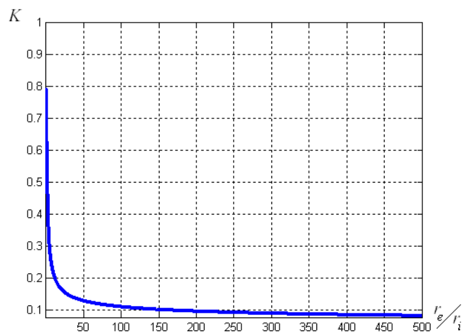

Fig.5. $K=K(r_e/r_i)$ for $[0<r_e/r_i<500]$.

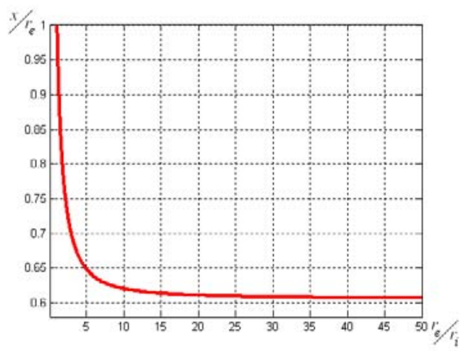

Fig.6. Equivalent radius x for $[0<r_e/r_i<50]$.

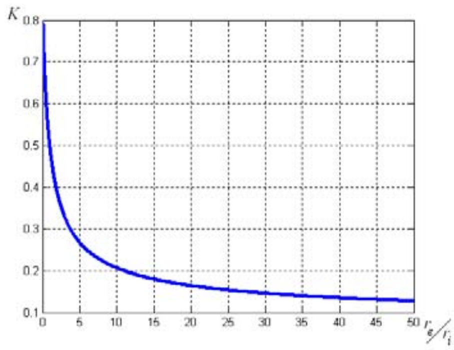

Fig.7. K=K(r_e/r_i) for [0<r_e/r_i<50].

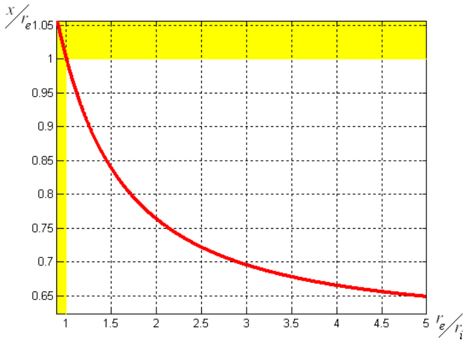

Fig.8. Equivalent radius x for [0<r_e/r_i<5].

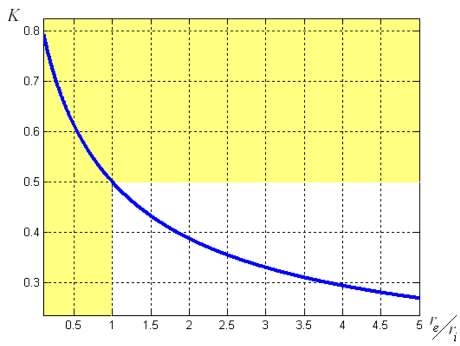

Fig.9. K=K(r_e/r_i) for [0<r_e/r_i<50].

Solution correctness is easily accomplished considering the limit case where the insulating cylinder is extremely thin, i.e. $r_e \to r_i$. In this limit case their ratio becomes unitary and the value for $r_i \le x \le r_e$ will lead to $x/r_e \to 1$ (according to Fig. 8). For such very thin coating it can be considered that all the points have the same temperature (by default $\theta_i = \theta_e = \theta$). According to Figure 9, we obtain $K = 0.5$ (the maximum value possible) and substituting it in equation (15) it results:

$$\theta_x = \theta_{se} + K \cdot (\theta_{si} - \theta_{se}) = \theta_{se} + 0{,}5 \cdot (\theta_{si} - \theta_{se}) = \\ = \frac{\theta_{si} + \theta_{se}}{2} = \theta \qquad (18)$$

The analysis of the curves in Figure 4 ÷ Figure 9 shows a rapid decrease of the ratio x/r_e from the unitary value to about 0.65 for thin tubular insulating materials (with the ratio $r_e/r_i < 5$). Further increase of the thickness of the insulating material leads to a slow decrease of the ratio x/r_e , which tends to stabilize at a value less over 0.6.

Parameter K has a relatively similar evolution, except that at high values of r_e/r_i its speed of decrease is more pronounced. A synthetic representation of the above conclusions can be followed in Table I.

TABLE I.
x/r_e AND K VALUE RELATIVE TO r_e/r_i

r_e/r_i	x/r_e	K
1	1	0.5
2	0.7642	0.3880
3	0.6958	0.3301
5	0.6486	0.2690
10	0.6208	0.2070
20	0.6111	0.1644
50	0.6075	0.1274
100	0.6068	0.1085
200	0.6066	0.0943
500	0.6065	0.0805

CONCLUSION

Generally, an equivalent mathematical relations system for describing a physical system can be achieved if first we establish a clear criterion which represents the base of the equivalence.

In this case the criterion was "the equivalence in terms of the energy stored” in the material. The obtained relations will be used for the simulation, using Matlab-Simulink toolkit, of some electro-thermal systems with cylindrical symmetry. Implementation of the formulas (15), (16) and (17) allows adapting the modeling system shown in Figure 1 and Figure 2 for the multi-layer walls with cylindrical symmetry.

Received on June 01, 2015
Editorial Approval on November 15, 2015

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