Analysis of the Phaseportrait for the Lü Dynamical System in a Particular Case

Adela Ionescu*, Florian Munteanu†
* University of Craiova/Department of Applied Mathematics, Craiova, Romania, adela0404@yahoo.com
† University of Craiova/Department of Applied Mathematics, Craiova, Romania, munteanufm@gmail.com

***Abstract* - In the present paper it is started a qualitative analysis of the Lü dynamical system, using the appropiate tools of Hamilton-Poisson geometry and using the software for mathematics MAPLE 15. The Lü's system arise from electrical engineering networks and it is very known that he has a chaotic behavior. For this reason, the problem of finding the solution of the system could be very difficult. By obtaining of a geometric Hamilton-Poisson structure, we can find such a solution as the intersection of two surfaces, the surfaces equation being given by the Hamiltonian H and the Casimir function K. Using MAPLE 15, it will be analyzed the phase portrait for two particular simplified versions of the dynamical system associated to the Lü ordinary differential system (a=0, b=c=0 and a=1, b=c=0). Also, we will do a study of the Lyapunov stability of Lü's system for the particular case (a=1, b=c=0). We obtain that the origin is an unstable equilibrium point for this particular parameter case of the Lü's model. This fact is confirmed by the numerical simulations. More that, the pictures show that the origin is a non-stable focus, in the first simulation case. Analytical results are accompanied by numerical illustrations.**

Cuvinte cheie: *system dynamic Lü, structura Hamilton-Poissone, portretul fazelor stabilitate Lyapunov.*

Keywords: *Lü dynamical system, Hamilton-Poisson structure, phaseportrait, Lyapunov stability.*

I. INTRODUCTION

The chaotic oscillations in a physical system, that is the chaotic behavior of the system, was observed in practical applications of many fields, from engineering to biology and economics. Using the tools of dynamical system theory, the Chaos can be suppressed using linear or nonlinear feedback methods ([1], [2], [3], [4]). Then, using a simple linear controller, the system is driven to a stable state ([5], [6]). The Lyapunov function method ([7]) is very used in this purpose and, also, the tools of numerical analysis are very useful to support the analytical results ([8], [9]).

The history of the study of RLC networks starts not very far, from the early fifties, but the technical explanations, journal references, and interactions with other disciplines are rich ([10], [11]). This theory lies basically on the conceptual modeling of electrical circuits. It is advantageous to realize the description of a system in terms of an ideal model, like an interconnection of idealized elements. These idealized elements (which are simple models) are used to approximate the properties of separate physical elements of the system.

The developments of linear network theory focused the attention of researchers. Besides the various fundamental contributions, there has always been a particular interest in studying the relations between basic equations of mechanics and the basic laws of electrical networks.

For a quite long time, the results of classical mechanics were the basis of the mathematical modeling of linear networks, namely, the models have been stated basically in terms of the Lagrangian and Hamiltonian formulations of physical systems ([12], [13], [14]). These formulations are important because they provide a systematic, compact and elegant system description in which physical quantities like energy, interconnection and dissipation play a central role. Thus, the results may be easily translated into network terminology.

The Lü's dynamical system was first proposed by J. Lü and G. Chen in 2002 in the paper [15]. This system is a model of a nonlinear electrical circuit, and we want to study it from mechanical geometry point of view and to point out some of its geometrical and dynamical properties. The original Lü system has the following form ([15], [16]):

$$\begin{cases} \dot{x} &= a(y-x) \\ \dot{y} &= -xz+by \\ \dot{z} &= -cz+xy \end{cases} \tag{1}$$

where a, b, c are real parameters.

The Lü dynamical system (1) admits a Hamilton-Poisson realization for some values of the parameters a, b, and c ([17], [18], [19]). More exactly, the Lü system admits Hamilton-Poisson realization with a three degree polynomial function as the Hamiltonian only if $a=b=c=0$, or $a=0,\ b=c$.

The Hamilton-Poisson realization offers us the tools to study the Lü system from mechanical geometry point of view.

The Lü system has three unknown uncertain parameters. It was studied from various standpoints. The system exhibits a chaotic attractor at the parameter values $a=36$, $b=3$ and $c=20$ ([3], [16]). Taking into account these values, a projective synchronization of two identical Lü attractors was realized by an adaptive feedback control. The analysis of the errors' trajectories for two identical Lü systems with adaptive feedback control issued useful applications ([3], [12], [15]). This autonomous system of ordinary differential equations, together with Lorentz's system and Chua's system, was generally accepted then as having chaotic behavior ([4], [10], [11], [12]).

The first aim of the present paper is to test the Lü dynamical system behavior for two sets of the parameters: $\{a=b=c=0\}$ and $\{a=1, b=c=0\}$. The soft Maple

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was used to analyze the phase portrait of the model is this cases. The second aim is to analyze the existence of a Lyapunov function for a particular case of the Lü system in order to check the stability or instability of the equilibrium.

II. A Brief Review of The Hamilton-Poisson geometry

In this section it will be present briefly the main notions from the geometry of Hamilton-Poisson structures (like in [12] and [16]).

If M is a smooth manifold and $C^\infty(M)$ denote the set of the smooth real functions on M, then *a Poisson bracket* (or *a Poisson structure*) on M is a bilinear map from $C^\infty(M)\times C^\infty(M)$ into $C^\infty(M)$, denoted by

$$(F,G)\to\{F,G\}\in C^\infty(M),\ \forall\, F,G\in C^\infty(M)$$

which verifies the following properties:

for all $F,G,H\in C^\infty(M)$, we have

Skew-symmetry:

$$\{F,G\}=-\{G,F\}$$

Jacobi identity:

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0$$

Leibniz rule:

$$\{F,G\cdot H\}=\{F,G\}\cdot H+G\cdot\{F,H\}$$

The pair $(M,\{\cdot,\cdot\})$ is called *a Poisson manifold*.

If $\{\cdot,\cdot\}$ is a Poisson structure on $\boldsymbol{R}^n$, then for any $f,g\in C^\infty(\boldsymbol{R}^n)$ we have the following relation:

$$\{f,g\}=\sum_{i,j=1}^{n}\{x_i,x_j\}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

Any Poisson structure on $\boldsymbol{R}^n$ is completely determined by the matrix $\Pi=\left(\{x_i,x_j\}\right)_{i,j=\overline{1,n}}$ via the relation:

$$\{f,g\}=(\nabla f)\Pi(\nabla g)^t$$

A Hamilton-Poisson system on $\boldsymbol{R}^n$ is a triple

$(\boldsymbol{R}^n,\{\cdot,\cdot\},H)$, where $\{\cdot,\cdot\}$ is a Poisson bracket on $\boldsymbol{R}^n$ and $H\in C^\infty(\boldsymbol{R}^n)$ is called *the energy* or *the Hamiltonian*. Its dynamics is described by the following system of differential equations:

$$\dot{x}=\Pi(\nabla \mathrm{H})^t$$

where $\dot{x}=(\dot{x}_1,\dot{x}_2,\ \dots,\ \dot{x}_n)^t$.

A Casimir function for the Poisson structure $\{\cdot,\cdot\}$ on $\boldsymbol{R}^n$ is a function $K\in C^\infty(\boldsymbol{R}^n)$ which satisfies

$$\{f,K\}=0,\ \text{for all } f\in C^\infty(\boldsymbol{R}^n).$$

A function $F\in C^\infty(\boldsymbol{R}^n)$ is called *conservation law* (or *first integral*) for Hamilton–Poisson system $(\boldsymbol{R}^n,\{\cdot,\cdot\},H)$, if the total derivative of F, vanishes, i.e.

$$\dot{F}=\frac{dF}{dt}=\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}\dot{x}_i=\sum_{i,j=1}^{n}\{x_i,x_j\}\frac{\partial F}{\partial x_i}\frac{\partial H}{\partial x_j}=0$$

or,

$$\{F,H\}=0.$$

It is known that any Hamilton-Poisson system who admit a Casimir function has at least two conservation laws: the Hamiltonian H and the Casimir function K. Also, any linear combination of H and K, $F_{a,b}=aH+bK$, $a,b\in\mathbf{R}$, is a new conservation law for and this first integrals H, K, $F_{a,b}$ are linear dependent ([12], [16], [19], [20]).

If a dynamical system has a Hamilton-Poisson realization, that means it can be written in the form $\dot{x}=\Pi(\nabla\mathrm{H})^t$. In this case, the dynamical system has not a chaotic behavior and the problem of finding conservation laws is very important for the study of the integrability and stability of the system ([14], [15], [16], [19], [20], [21]).

III. The hamilton-poisson Realizations for Simplified versions of the Lü System

In this section it will be presents some Hamilton-Poisson structures of the dynamical system of Lü. This system who arise from electrical engineering networks has a chaotic behavior. For this reason, the problem of finding the solution of the system could be very difficult.

A Hamilton-Poisson realization offers us the possibility to find this solution as the intersection of two surfaces, the surfaces equation being given by the Hamiltonian H and the Casimir function K.

There are only four cases for which the Lü system admits as Hamiltonian a three degree polynomial function and finding another kind of function as a Hamiltonian of the Lü system remains an open problem ([15], [16], [17], [18]).

From [16] and [17], we have that:

I. For a non null and $\mathrm{b}=\mathrm{c}=0$ the system becomes:

$$\begin{cases}\dot{x} &= a(y-x)\\ \dot{y} &= -xz\\ \dot{z} &= xy\end{cases}\qquad(2)$$

The function

$$H(x,y,z)=\alpha(x^2+y^2)+\beta,$$

where $\alpha,\beta\in\boldsymbol{R}$, is a constant of motion (conservation law) for this system.

II. For $\mathrm{a}=0$ and b,c are non null real numbers the system becomes:

$$\begin{cases}\dot{x} &= 0\\ \dot{y} &= -xz+by\\ \dot{z} &= -cz+xy\end{cases}\qquad(3)$$

The function

$$H(x,y,z=\ \mathrm{f(x)},$$

where $f\in C^1(\boldsymbol{R})$, is a constant of motion (conservation law) for this system.

III. For a non null and $\mathrm{b}=\mathrm{c}$ any real numbers the system becomes:

$$\begin{cases}\dot{x} &= a(y-x)\\ \dot{y} &= -xz+by\\ \dot{z} &= -bz+xy\end{cases}\qquad(4)$$

The function

$$H(x,y,z)=\alpha(\mathrm{xy^2}-2\mathrm{byz}+\mathrm{xz^2})+\mathrm{f(x)},$$

where where $\alpha\in\boldsymbol{R}$ and $f\in C^1(\boldsymbol{R})$, is a constant of motion (conservation law) for this system.

IV. For $a = b = c = 0$ the system becomes:

$$\begin{cases} \dot{x} & = & 0 \\ \dot{y} & = & -xz \\ \dot{z} & = & xy \end{cases} \qquad (5)$$

The function

$$H(x,y,z) = \alpha(x^2 + y^2) + \beta(xy^2 + xz^2) + f(x),$$

where α, $\beta \in \boldsymbol{R}$ and $f \in C^1(\boldsymbol{R})$, is a constant of motion (conservation law) for this system.

For this, it is enough to check that $\mathrm{dH} = 0$ for each case mentioned above, that means that the function H is constant along the integral curves of the dynamical system.

The first case is when a is any real number and $\mathrm{b} = \mathrm{c} = 0$. For this specific case, there exists a Hamilton-Poisson realization if and only if $\mathrm{a} = 0$ ([17]).

For the case $\mathrm{a} = 0$ and b, c are any real numbers, in [17] it has proved that the Hamilton-Poisson realization exists only if $\mathrm{b} = \mathrm{c}$.

The last two cases, $\mathrm{a} = 0$, $\mathrm{b} = \mathrm{c}$ and $\mathrm{a} = \mathrm{b} = \mathrm{c} = 0$, can be studied as the first two cases.

We can conclude that the Lü system admits Hamilton-Poisson realization with a degree three polynomial function as the Hamiltonian only if $\mathrm{a} = \mathrm{b} = \mathrm{c} = 0$, or $\mathrm{a} = 0$, $\mathrm{b} = \mathrm{c}$.

IV. Phaseportrait of the Lü Model for Some Testing Parameters Values

The aim of the present paper is to test the Lü dynamical system behavior for some starting values of the parameters set. The sets $\{\mathrm{a} = \mathrm{b} = \mathrm{c} = 0\}$ and $\{\mathrm{a} = 1, \mathrm{b} = \mathrm{c} = 0\}$ are the simplest studied ([16], [17]). The soft Maple was used to analyze the phase portrait of the model ([9], [16]). The flexible appliance "phaseportrait" allows the interactively use of all options.

For the present aim the $[\mathrm{x(t)}, \mathrm{y(t)}]$ phaseportrait was chosen. Two cases for the discrete time were taken into account, and also two initial conditions sets:

$$\{\mathrm{x}(0) = 1, \mathrm{y}(0) = 0, \mathrm{z}(0) = 1\} \quad \text{and}$$

$$\{x(1) = 0, y(1) = 1, z(1) = 1\}$$

Thus, the following simulation cases were analyzed:

Ai) $\mathrm{a} = 1, \mathrm{b} = \mathrm{c} = 0$, $\{\mathrm{x(t)} = 1, \mathrm{y(t)} = 0, \mathrm{z(t)} = 1\}$;

Aii) $\mathrm{a} = 1, \mathrm{b} = \mathrm{c} = 0$, $\{\mathrm{x(t)} = 0, \mathrm{y(t)} = 1, \mathrm{z(t)} = 1\}$;

Bi) $\mathrm{a} = \mathrm{b} = \mathrm{c} = 0$, $\{\mathrm{x(t)} = 1, \mathrm{y(t)} = 0, \mathrm{z(t)} = 1\}$;

Bii) $\mathrm{a} = \mathrm{b} = \mathrm{c} = 0$, $\{\mathrm{x(t)} = 0, \mathrm{y(t)} = 1, \mathrm{z(t)} = 1\}$;

Each simulation case was analyzed with the discrete time units $\mathrm{t} = 50$ and $\mathrm{t} = 1\,00$. It was observed that when doubling the time, the values strip does not change in all cases for the trajectory. The figures are as follows. Some features are labeled on each figure.

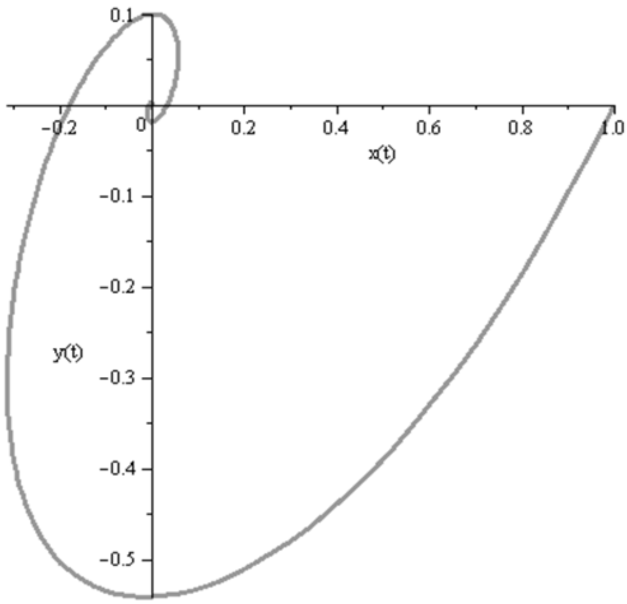

Fig. 1. Case Ai, t = 0 ... 45

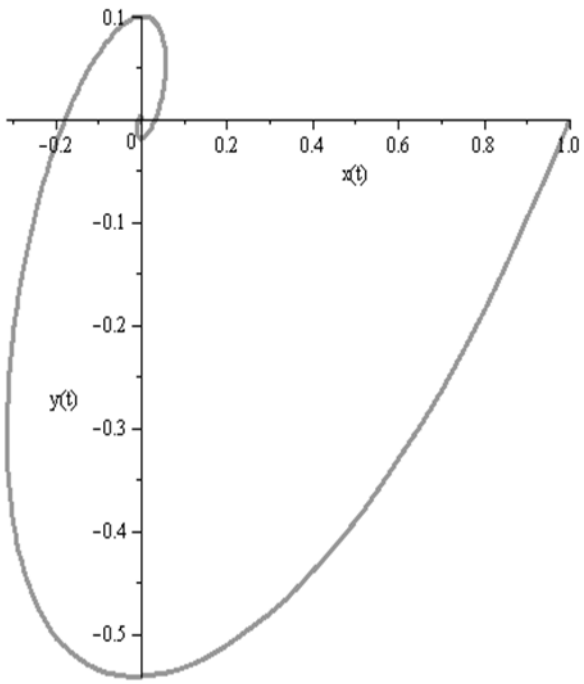

Fig. 2. Case Ai, t = 0 ... 100. The trajectory doesn't change the trend

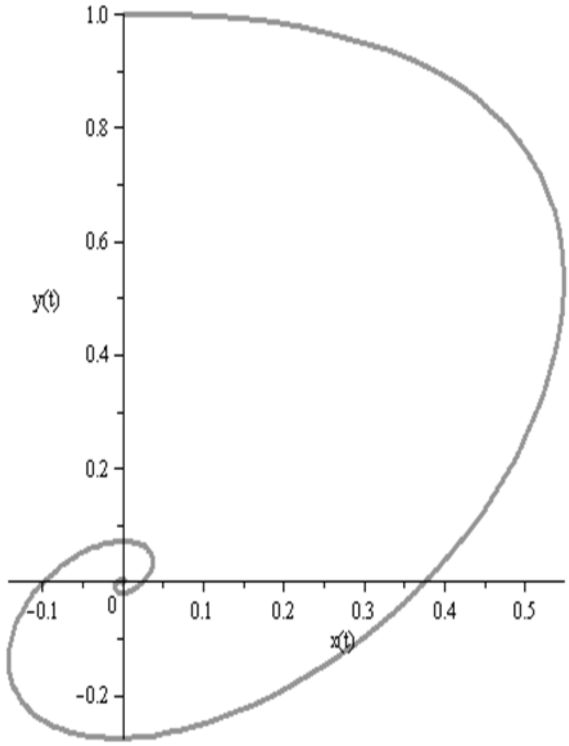

Fig. 3. Case Aii, t = 0 … 45. The trajectory becomes positive

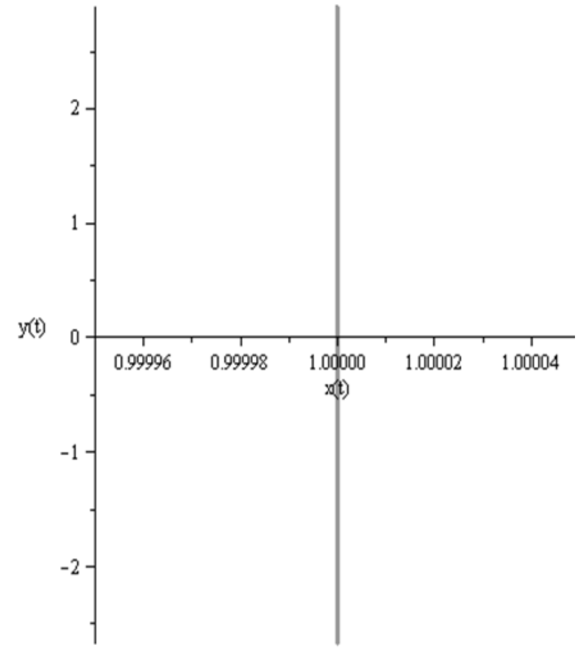

Fig. 4. Case Bi, t = 0 … 45. The movement is only along the x axis

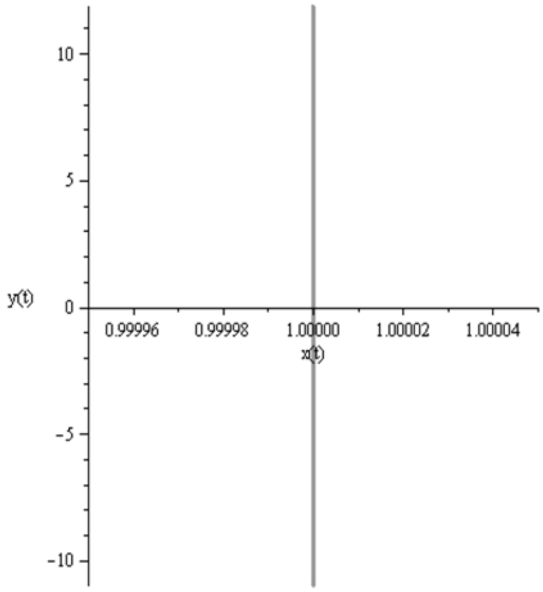

Fig. 5. Case Bi, t = 0 … 100

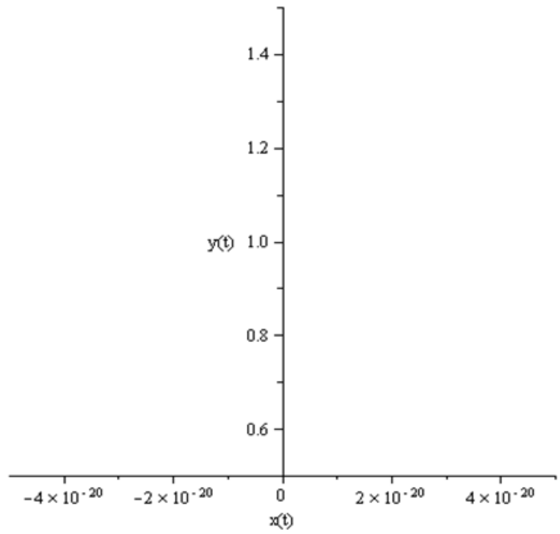

Fig. 6. Case Bii, t = 0 ... 45. The trajectory cannot be seen, the interval on x axis has infinitely small values

V. LYAPUNOV STABILITY FOR THE LÜ MODEL IN THE SIMPLIFIED VERSION

Let us consider the Lű system in the simplified form presented in the above section (a=1, b=c=0):

$$\begin{cases} \dot{x} &= y - x \\ \dot{y} &= -xz \\ \dot{z} &= xy \end{cases} \tag{6}$$

It is immediate that the origin $\mathrm{O} = (0,0,0)$ is an equilibrium point.

In this section we analyze the existence of a Lyapunov function for the system (6) in order to check the stability/instability of the equilibrium. There are few criteria for analyzing the stability, very used in the literature. We shall test here *the first (reduced) criterion* and *the Krasovskii theorem* ([7]).

First Lyapunov criterion (*reduced method*): the stability analysis of an equilibrium point x_0 is done studying the stability of the corresponding linearized system in the vicinity of the equilibrium point.

In the neighborhood of an equilibrium point (x_0, u_0), the following statements hold:

A. If all the eigenvalues of matrix linearized matrix $\widetilde{\mathrm{A}}$ of the system have "negative real part", then the equilibrium point (x_0, u_0) is asymptotically stable also for the nonlinear system;

B. If at least one of the eigenvalues of the matrix $\widetilde{\mathrm{A}}$ has "positive real part", then the equilibrium point (x_0, u_0) is unstable also for the nonlinear system.

C. If at least one eigenvalue of the matrix $\widetilde{\mathrm{A}}$ is located "on the imaginary axis" while all the other eigenvalues have "negative real part", then it is not possible to conclude anything about the stability of the equilibrium point (x_0, u_0) for the nonlinear system. In this case the criterion is not effective.

The linearized matrix $\widetilde{\mathrm{A}}$ associated to the system (6), that is the Jacobian J, is the following:

$$\tilde{A} = \begin{pmatrix} -1 & 1 & 0 \\ -z & 0 & -x \\ y & x & 0 \end{pmatrix} \tag{7}$$

Then one can easily observe that the characteristic equation in origin is

$$\lambda^2 (1 + \lambda) = 0$$

with real roots. Therefore, *the reduced Lyapunov criterion cannot be applied*.

Taking into account that the system (6) is non linear, we apply the Krasovskii theorem ([7]) as follows.

Let be J from (7) the Jacobian associated to the system (6). If $\mathrm{J} + \mathrm{J}^{\mathrm{T}}$ is negative defined, then

$$\mathrm{V} = f^{\mathrm{T}}(\mathrm{x}) * \mathrm{f}(\mathrm{x})$$

is a Lyapunov function for the system (6) and the equilibrium is asymptotically stable.

Here f(x) denotes the right side of the system.

We have the following relation for the Jacobian associated to (6)

$$\mathrm{J} = \begin{bmatrix} -1 & 1 & 0 \\ -z & 0 & -x \\ y & x & 0 \end{bmatrix} \tag{8}$$

Then we have

$$\mathrm{J} + \mathrm{J}^{\mathrm{T}} = \begin{bmatrix} -2 & 1-z & y \\ 1-z & 0 & 0 \\ y & 0 & 0 \end{bmatrix} \tag{9}$$

One observes that $\Delta_1 = -2 < 0, \Delta_2 = \Delta_3 = 0$.

Therefore, we cannot define a Lyapunov function with the above definition. Thus, *the origin is asymptotically unstable equilibrium* for the Lű model in this simplified form. This fact is confirmed by the numerical simulations from the previous section. In the previous section, the pictures show that the origin is a non-stable focus, in the first simulation case.

VI. CONCLUSIONS

In the present note it is started a qualitative analysis of the Lü dynamical system. For the moment, the simplest sets for the parameters values were chosen as study case.

It is immediate to see that a slightly difference in the parameter values in the cases A and B, produces an important change in the trajectory trend. In the case A of simulation, the origin is a solution, moreover it is a focus which changes the sign when changing the initial conditions. Doubling the time units produces no change in the values limits on x and y. Therefore we can have an important notice, that the movement remains on the same trajectory, for any large time units' number.

In the second case for all parameters zero, we have in fact a constant motion on the x axis. Doubling the time units produces only an increasing of the limits on y axis. This constant motion is confirmed by the constant conservation law found geometrically. When changing the initial conditions it issues an flagrant change of the trajectory: in fact the variation interval on x axis is extremely small and the trajectory is no more visible.

Let us notice that the time units in Maple are non-dimensional, the user can choose this dimension function of the scientific purposes. This flexible feature of graphical appliances allows their use in all types of simulations.

The significant difference between the two simulation cases confirms that we have a model which is sensitive to initial conditions. This enables us to consider a next target, finding a state-space linearization by feedback linearization technique. Finding the equivalent model of the Lü dynamical system, would help to better study the influence of the parameters on the model behavior and also to realize further qualitative analysis.

The fact that the origin is an unstable equilibrium point for this particular parameter case of the model, justifies the approach of controlling the chaos in the Lü dynamical system, in its general form [5], and to further analyze the stability for different parameters.

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