Finite Volume Method for Electrostatic Problems

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Abstract - **In this paper, we examine the problem of determinining the electrostatic potential distribution and field intensity vector in the high voltage divider and around the power line support. The potential is defined as the solution of the Dirichlet problem for the Poisson equation, and the flow of the intensity vector is defined by integration of this vector along the contour located within the solution domain. The formulated problem is solved numerically by means of the finite volume method. This method representes some generalization of the finite difference method and allows discretization of differential equations on grids with arbitrary configuration. The idea of the method is to construct a basic grid, consisting of triangles, and the dual grid, consisting of the Voronoi cells. The differential equations are integrated over the volume of the Voronoi cell and then, using the divergence theorem, the volume integrals are replaced by surface integrals. The integrals over the cell surface are approximated by quadrature formulas. As a result, the original differential equation is replaced by a difference equation. This procedure is performed for all internal nodes, and therefore we obtain a system of linear algebraic equations. The technique was applied for solving two practically important problems. The fields of potential and of flow intensity vector have been constructed for the problem of determining the electrostatic field in the high-voltage divider. The divider capacity with the screen and without screen was determined. It was shown that the use of a cylindrical screen results in an almost twofold increase in capacity. The second considered problem is related to the calculation of the electrostatic field in the vicinity of the L-shaped support of the power line.**

Keywords - *finite volume method, electrostatic potential, field intensity vector, capacity, Dirichlet problem, Poisson equation*

I. INTRODUCTION

Let consider the problem of determination of potential distribution $u(x,y,z)$ of electrostatic field in the multiply-connected domain Ω with piecewise constant permittivity $\varepsilon_a(x,y,z)$. Within the Ω, the function $u(x,y,z)$ satisfies Poisson equation

$$\mathrm{div}(\varepsilon_a \mathrm{grad} u) = -\sigma(x,y,z) \quad (1)$$

where $\sigma(x,y,z)$ is the density of free charge distribution. If within Ω there are no any of such charges, then the equation (1) turns into Laplace equation $\mathrm{div}(\varepsilon_a \mathrm{grad} u) = 0$. The values of $u(x,y,z)$ on the boundary $\Gamma = \partial\Omega$ of the Ω are known

$$u(x,y,z)\big|_{\Gamma} = \mu(x,y,z) \quad (2)$$

The electric field intensity (also called electric field) $\vec{E}$ is defined by the formula $\vec{E} = -\mathrm{grad}\, u$ and the electric displacement field – by the formula $\vec{D} = \varepsilon_a \vec{E}$. On the boundary interfaces between the heterogeneous mediums the continuity conditions $[u] = 0$ and $[(\vec{D},\vec{n})] = 0$ hold. Here the square brackets denote the difference between the limit values at the left and at the right of the boundary interface, is the normal vector to this interface.

II. DISCRETE MODEL COMPOSITION

For numerical solving of the formulated Dirichlet's problem we divide the volumetric domain $\overline{\Omega} = \Omega + \Gamma$ into the finite set of small volumetric tetrahedron-shaped elements (pyramids). The vertices of pyramids are called the nodes of the difference grid. It is possible to construct a lot of various divisions of three dimensional domain into pyramids in case when the nodes have fixed positions. The division that is known as Delaunay triangulation is considered the best one. Delaunay triangulation is a division such that no other node of the grid is inside of the circumsphere of any pyramid.

Let denote by T_h the set of grid pyramids, where h is the maximal value of the pyramids side lengths. Let introduce also the dual grid T_h^* that consists of so-called Voronoi cells. Each Voronoi cell encloses one of the inside nodes of the difference grid. The example of three-dimensional Voronoi cell for basic node number 74 is represented in the Fig. 1. One can observe that this basic node is connected with the nodes 2, 73, 75, 80, 176, 182, 188 and 332, which are called neighboring nodes, but the Voronoi cell represents the polyhedron with semitransparent faces. Each face of the cell is orthogonal to the segment between the basic node and neighboring node and the intersection point between the face and the segment is situated at the midpoint of the segment.

Let denote by P_0 the basic node and by $K_{P_0}^*$ – the Voronoi cell. The vertices of Voronoi cell $K_{P_0}^*$ we denote by Q_i. These vertices Q_i are the centers of the spheres circumscribed around the tetrahedrons having the point P_0 as a vertex.

As an approximate solution of the problem (1), (2) we consider the piecewise linear function that must be continuous in $\overline{\Omega}$ and linear on every tetrahedron $K \in T_h$. The function $u_h(x,y,z)$ on the set of tetrahedron T_h can be defined in the following manner.

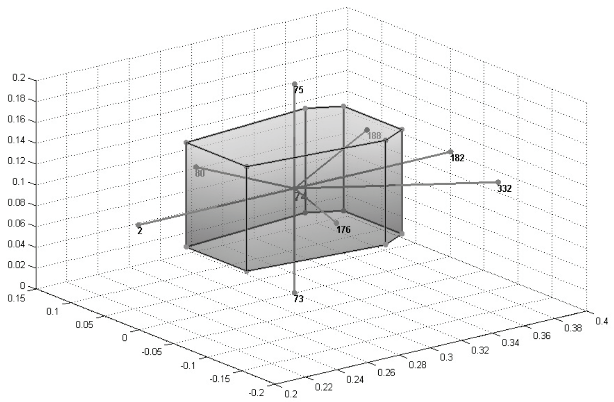

Fig. 1. Voronoi cell for basic node number 74.

Let the tetrahedron $K=\overline{P_1P_2P_3P_4}$ be some element of the set T_h and $P(x,y,z)$ be an arbitrary point of this element. In this tetrahedron for each vertex we introduce the shape functions $N_i(x,y,z),\ i=\overline{1,4}$. These functions should verify the following conditions: the functions are linear and their values at the tetrahedron vertices are equal to 0 or 1, i.e. $N_i(P_k)=\begin{cases}1, & i=k\\ 0, & i\neq k\end{cases}$. The shape functions can be represented in the explicit form through the coordinates of the vertices:

$$N_i(x,y,z)=w_{1,i}x+w_{2,i}y+w_{3,i}z+w_{4,i} \qquad (3)$$

Here $w_{1,i}$, $w_{2,i}$, $w_{3,i}$ and $w_{4,i}$ are the components of the vectors $\overline{w}_i$, $i=\overline{1,4}$. To determine the vectors $\overline{w}_i$ it is necessary to solve 4 systems of equations $A\overline{w}_i=\bar{f}_i$, $i=\overline{1,4}$. The elements of the matrix A are formed from the coordinates of the vertices $P_i=P_i(x_i,y_i,z_i),\ i=\overline{1,4}$ of tetrahedron as follows

$$A=\begin{pmatrix}x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\\ x_4 & y_4 & z_4 & 1\end{pmatrix};\ \bar{f}_i=(f_{1,i},f_{2,i},f_{3,i},f_{4,i}),\ f_{k,i}=\begin{cases}1, & i=k\\ 0, & i\neq k\end{cases}$$

Using the shape functions for every grid node (internal or boundary) we introduce the basis function $\varphi_i(x,y,z)$, i = 1, 2, …, n, n + 1, …, n_1 (n and n_1 represent here the number of internal nodes and the total number of nodes correspondingly). The function $\varphi_i(x,y,z)$ is piecewise linear, i.e. it is continuous and linear on each tetrahedron with unit value in the node P_i and with zero values in all other nodes. Then the approximate solution $u_h(x,y,z)$ can be represented as a linear combination of basis functions

$$u_h(x,y,z)=\sum_{i=1}^{n_1}u_i\varphi_i(x,y,z) \qquad (4)$$

It is easy to verify that the coefficients u_i from (4) are equal to the unknown potential values at the node $P_i(x_i,y_i,z_i)$, i.e. $u_h(x_i,y_i,z_i)=u_i$.

It should be noted that the solution of the problem (1), (2) by finite element method requires the application of Galerkin method. This method consists in following. Let substitute the (4) in the equation (1) and then write down the condition of orthogonality of obtained expression with respect to basis functions $\varphi_k(x,y,z)$ for internal nodes:

$$\int_\Omega \mathrm{div}(\varepsilon_a \mathrm{grad} u_h)\varphi_k dV=-\int_\Omega \sigma\varphi_k dV,\ k=\overline{1,n} \qquad (5)$$

$$\int_\Omega \mathrm{div}(\varepsilon_a \mathrm{grad}\sum_{i=1}^{n_1}u_i\varphi_i)\varphi_k dV=\sum_{i=1}^{n_1}\alpha_{ki}u_i=\tilde{\beta}_k \qquad (6)$$

$$\tilde{\beta}_k=-\int_\Omega \sigma\varphi_k dV$$

$$\alpha_{ki}=\int_\Omega \mathrm{div}(\varepsilon_a \mathrm{grad}\varphi_i)\varphi_k dV=-\int_\Omega \varepsilon_a \mathrm{grad}\varphi_i \mathrm{grad}\varphi_k dV$$

As the solution values are known at the boundary nodes, then the system (6) takes the form

$$\sum_{i=1}^{n}\alpha_{ki}u_i=\beta_k.\ \beta_k=\tilde{\beta}_k-\sum_{i=n+1}^{n_1}\alpha_{ki}\mu_i,\ k=\overline{1,n} \qquad (7)$$

In contrast to finite element method, the generalized Galerkin method is used in finite volume method. This generalized approach consists in following. In the condition of orthogonality (5) we use basis functions $\psi_k(x,y,z)$ of the space $W_2^0(\Omega)=L_2(\Omega)$ as follows [1, 2]. Let introduce new basis functions $\psi_k(x,y,z)$ for dual grid T_h^* by the following rule: function $\psi_k(x,y,z)$ possesses the constant unit values in the Voronoi cell for internal node P_k and it possesses zero values in the rest of domain. Then the condition of orthogonality (5) with functions $\psi_k(x,y,z)$ gets the form

$$\int_\Omega \mathrm{div}(\varepsilon_a \mathrm{grad} u_h)\psi_k dV=-\int_\Omega \sigma\psi_k dV,\ k=\overline{1,n} \qquad (8)$$

Taking into consideration that the function $\psi_k(x,y,z)$ is nonzero only in $K_{P_k}^*$, we obtain

$$\int_{K_{P_k}^*}\mathrm{div}(\varepsilon_a \mathrm{grad} u_h)dV=-\int_{K_{P_k}^*}\sigma\, dV \qquad (9)$$

where $K_{P_k}^*$ is Voronoi cell for node P_k.

Thus, to obtain the system of linear algebraic equations with respect to unknown values of the function u_h at the grid nodes by means of finite volume method we should proceed as follows. Let consider the Poisson equation $\mathrm{div}(\varepsilon_a \mathrm{grad} u)=-\sigma(x,y,z)$ in three-dimensional space with Cartesian coordinates $Oxyz$. Let integrate this equation over the volume of the cell $K_{P_k}^*$.

Then we obtain the formula coinciding with (9)

$$\int_{K^*_{P_k}} \mathrm{div}(\varepsilon_a \mathrm{grad} u) dV = -\int_{K^*_{P_k}} \sigma(x,y,z)\, dV \quad (10)$$

Now we apply the divergence theorem to the left-hand member of the (10) and obtain

$$\int_{K^*_{P_k}} \mathrm{div}(\varepsilon_a \mathrm{grad} u) dV = \int_{\partial K^*_{P_k}} \varepsilon_a (\mathrm{grad} u, \vec{n})\, dS = \int_{\partial K^*_{P_k}} \varepsilon_a \frac{\partial u}{\partial n} dS \quad (11)$$

where $\partial K^*_{P_k}$ is the total surface of the polyhedron $K^*_{P_k}$; $\vec{n}$ is the external normal to the surface $\partial K^*_{P_k}$, and $\partial u / \partial n$ is the derivative of function u by this normal. In this case the equation (10) takes the form

$$\int_{\partial K^*_{P_k}} \varepsilon_a \frac{\partial u}{\partial n} dS = -\int_{K^*_{P_k}} \sigma(x,y,z) d\tilde{V} \quad (12)$$

Thus, the solution of the problem (1), (2) by finite volume method reduces to the approximation of the relation (12) for Voronoi cells for internal nodes of the difference grid. The analogous procedure is proper to the finite difference method for grids with parallelepiped-shaped cells. Therefore, the finite volume method can be considered as some generalization of the finite difference method for block discretization with arbitrary shaped cells. By this reason the finite volume method keeps all advantages of the finite difference method. In comparison with finite element method the algorithm of finite-difference approximations here is not so sophisticated and we do not need to construct local and global stiffness matrices when forming the resolving system of equations of type (7).

Let's denote in the Voronoi cell $K^*_{P_0}$ by $P_i, i = \overline{0,8}$ the grid nodes; by $S_i, i = \overline{1,8}$ – the areas of the faces that are orthogonal with the segments $\overline{P_0 P_i}$; by $M_i, i = \overline{1,8}$ – the intersection points of the segment $\overline{P_0 P_i}$ and the face S_i. Then the integral from the formula (12) over the surface $\partial K^*_{P_0}$ we can approximate as follows:

$$\int_{\partial K^*_{P_0}} \varepsilon_a \frac{\partial u}{\partial n} dS = \sum_{i=1}^{8} \int_{S_i} \varepsilon_a \frac{\partial u}{\partial n} dS \cong \sum_{i=1}^{8} \varepsilon_a(M_i) \frac{u(P_i) - u(P_0)}{\left|\overline{P_0 P_i}\right|} S_i .$$

where $\left|\overline{P_0 P_i}\right|$ is the length of the segment $\overline{P_0 P_i}$.

The integral from the right-hand member of (12) we approximate by formula

$$\int_{K^*_{P_0}} \sigma(x,y) dV = \sigma(P_0) V_0$$

where V_0 is the volume of the Voronoi cell $K^*_{P_0}$. Then the approximation of the equation (12) can be represented in the following form

$$\sum_{i=1}^{8} \varepsilon_a(M_i) \frac{u(P_i) - u(P_0)}{\left|\overline{P_0 P_i}\right|} S_i = -\sigma(P_0) \tilde{V_0}$$

So the final equation for the grid node P_0 takes the following form

$$\alpha_0 u(P_0) + \sum_{i=1}^{8} \alpha_i u(P_i) = -\sigma(P_0) V_0 \quad (13)$$

$$\alpha_i = \varepsilon_a(M_i) \frac{S_i}{\left|\overline{P_0 P_i}\right|}, i = \overline{1,8}; \ \alpha_0 = -\sum_{i=1}^{8} \tilde{\alpha_i}$$

Now we can write out the equation in the form of (13) for each internal grid node and we use the condition (2) for the boundary nodes. As a result, we obtain the system of linear algebraic equations with symmetrical matrix. It is to mention that when solving the practically important problems the number of equations in such systems amounts to thousands or dozens of thousands. However, since each equation of the type (13) contains only some nonzero elements (usually there are from 9 to 25 non-zeros) then it turns out that the final matrix is sufficiently sparse matrix. In proposed algorithm only nonzero elements of the matrix are stored in computer memory. To solve the system we apply iterative conjugate gradient method that rapidly converges for problems of this type.

The obtained solution $u_h(x,y,z)$ for field potential distribution in $\overline{\Omega}$ permits to construct the flux of electric field intensity vector $\vec{E} = (E_x, E_y, E_z) = -\mathrm{grad}\, u$. Let denote by V the flux of vector $\vec{E}$ passing through the unit area element that is orthogonal with vector $\vec{E}$. Then the level curves $u(x,y,z) = \mathrm{const}$ and $V(x,y,z) = \mathrm{const}$ generate mutually orthogonal families. The function $V(x,y,z)$ can be obtained by calculation of the following contour integral

$$V(x,y) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \left(E_x dx + E_y dy + E_z dz\right) \quad (14)$$

where x_0, y_0, z_0 are the coordinates of an arbitrary fixed point from Ω and the patch of integration is situated inside of domain Ω. In case of multiply-connected domain the patch of integration also can not intersect the cuts of the domain that bring it to simply connected structure.

The capacitance C between two conducting bodies can be computed from the formula

$$C = \frac{q}{u_1 - u_2} \quad (15)$$

where $(u_1 - u_2)$ is potential difference of these bodies. The charge q of the body located inside of the some three-dimensional domain V can be computed in accordance with Gauss' law of flux as a surface integral of the field strength vector $\vec{E}$ over surface $S = \partial V$

$$q = \varepsilon \int_S \vec{E} \cdot d\vec{S} = -\varepsilon \int_S (\mathrm{grad} u \cdot \vec{n}) dS = -\varepsilon \int_S \frac{\partial u}{\partial n} dS \quad (16)$$

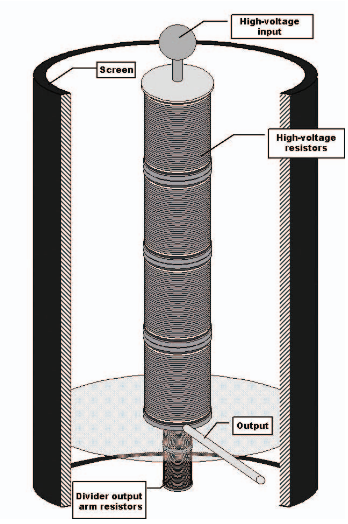

Fig. 2. General view of the high-voltage resistor with cylindrical screen.

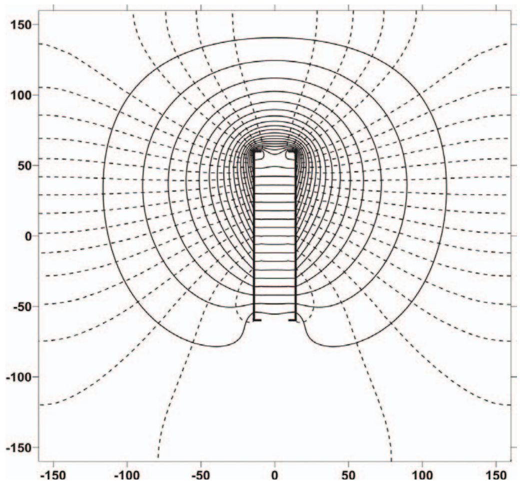

Fig. 3. Capacity of divider without screen C = 22.05 pF on the grid with 19675 nodes.

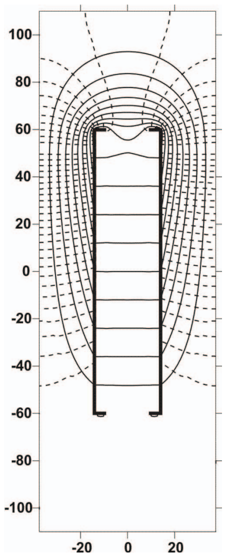

Fig. 4. Capacity of divider with cylindrical screen C = 40.99 pF on the grid with 66164 nodes.

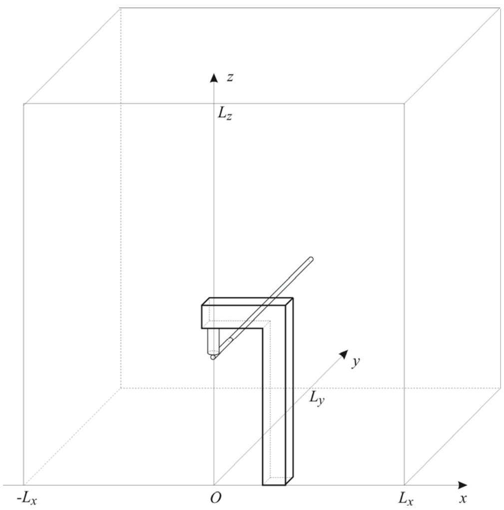

Fig. 5. L-shaped support with a hanging wire on a cylindrical glass isolator.

Here by S we denoted an arbitrary surface containing the charged body, by $\vec{n}$ – the exterior normal vector to the surface S and by ε – the permittivity.

III. Practical Implementation

The elaborated numerical algorithm is realized in the form of program system in Matlab development framework. The developed software has been used to solve two practical problems.

In engineering practice, it is often necessary to determine very precisely the capacitance of multiply connected piecewise homogeneous bodies, where the potential is known at the opened contours. The determination of the electrostatic fields and capacities of the high-voltage resistors (potential dividers), implemented on the base of microwire and protected by the conical or cylindrical screens (fig. 2), belongs to such problems with degenerated boundary conditions. Such a problem does not represent the classical Dirichlet's problem for simply connected or multiply connected domain since the boundary conditions are specified not only at the exterior boundary, but at the two broken lines within the domain of solution existence as well.

Since the considered problem possesses the axial symmetry property, then the domain of the solution is a two-dimensional rectangle in cylindrical coordinates r and z.

The Fig. 3 and Fig. 4 represent the potential and field strength level curves for typical constructions of the resistive divider with screen and without it. The resistor represents the hollow glass cylinder with the height H_1 = 120 mm, the external diameter D_1 = 28 mm and the internal diameter D_2 = 18 mm. The screen is of cylindrical form with the height H = 220 mm and the diameter D = 75 mm. The relative dielectric constant for glass is e_1 = 6, and it is e_2 = 1 for the air filling the internal and external frame hollows. The potential is given at the inner boundaries and it is linearly decreasing from 10 dimensionless units to zero.

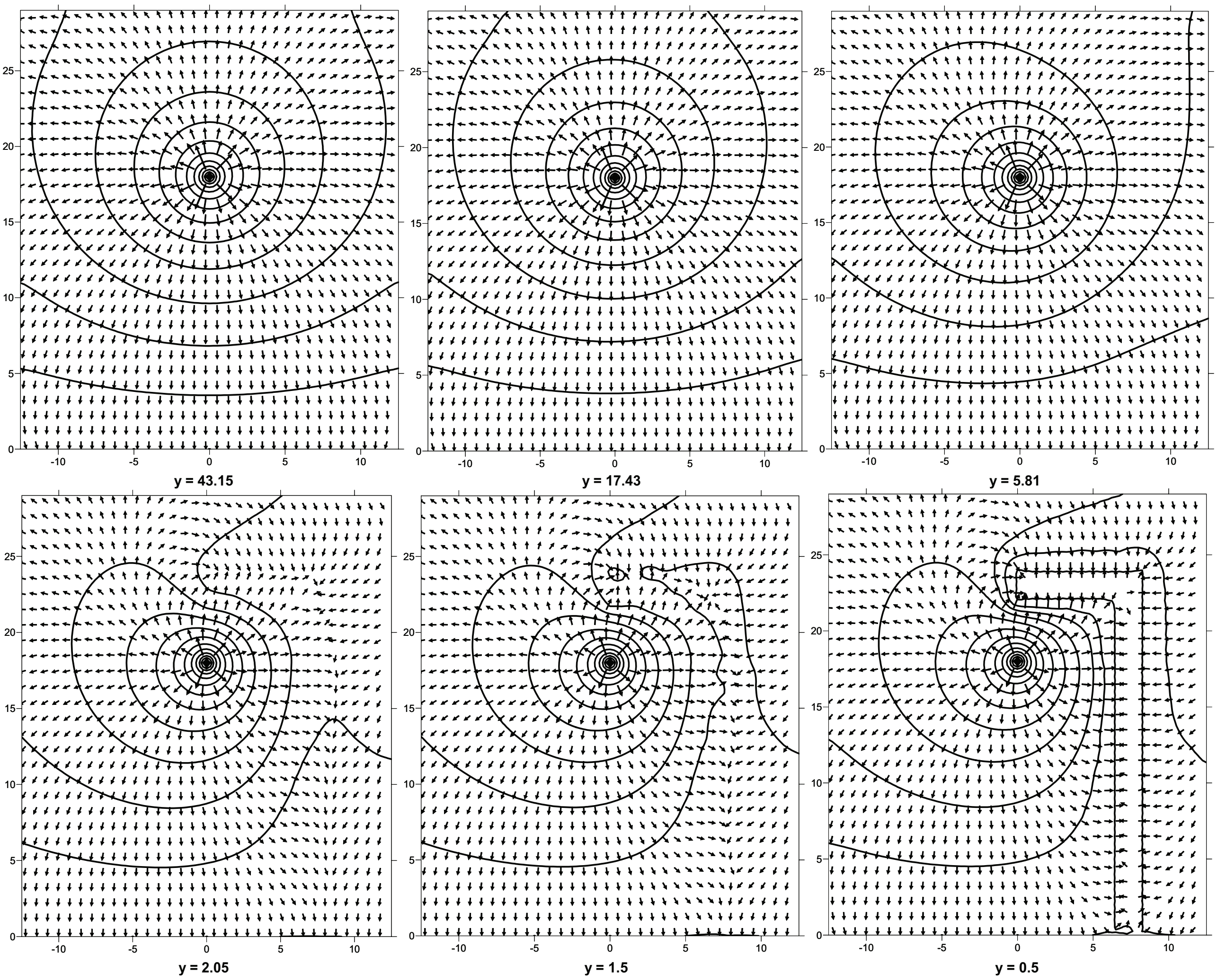

Fig. 6. Equipotential lines (full lines) and intensity vector (arrows) at the different cross-sections (y = 43,15; 17,43; 5,81; 2,05; 1,5 and 0,5 m).

The comparative analysis of the presented results shows that the presence of the screen with indicated dimensions approximately duplicates the electrical capacity of the divider.

The second considered problem is related to the calculation of the potential and the electrostatic field in the vicinity of the L-shaped support of the power transmission line with a voltage of 100 kV (Fig. 5).

Solution of this problem is presented in the Fig. 6.

The figure shows the level curves of the potential and the intensity vector at the cross-sections by the vertical planes y = 43,15; 17,43; 5,81; 2,05; 1,5 and 0,5 m.

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